CHARACTERIZATIONS OF α-WORDS, MOMENTS, AND DETERMINANTS

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1. INTRODUCTION

Throughout this paper we consider binary words. All results can easily be stated for words over other two-letter alphabets. For any word w, let $|w|$ denote the *length* of w and let $|w|_1$, called the *height* of w, denote the number of occurrences of the letter 1 in w. For $n \geq 1$ and $c_1, c_2, \ldots, c_n \in \{0,1\}$, define operators T and $\sim$ by

$$T(c_1c_2\ldots c_n) = c_2\ldots c_nc_1,$$

$$(c_1c_2\ldots c_n)^{\sim} = c_n\ldots c_2c_1.$$

For each integer j, let T^j have the obvious meaning. The operator T is called the *cyclic shift* (or *rotation*) *operator.* A word u is called a *conjugate* of a word w if $u = T^j(w)$ for some integer j. The set of all distinct conjugates of w is called the *conjugate class* of w and is denoted by $[w]$. The word $\tilde{w}$ is called the *reversal* of the word w.

A word w is said to be a *palindrome* if either w is the empty word or $\tilde{w} = w$. w is said to be *primitive* if it is not a power of another word. w is said to be a *Lyndon* (resp. *anti-Lyndon word* if it is the smallest (resp., largest) in the lexicographic order in the conjugate class of w. w is said to be *bordered* if there are words x and y with x nonempty such that $w = xyx$; otherwise, w is said to be *unbordered.*

For $w = c_1c_2\ldots c_q$, where each c_i is either 0 or 1, define $M(w) = \sum_{i=1}^{q}(q+1-i)c_i$. $M(w)$ is called the *moment* of w. Define

$$M([w]) = \{M(u) : u \in [w]\},$$
$$\delta(w) = \max\{M(u) - M(v) : u, v \in [w]\}.$$

One way to define α-words is to make use of T and the words $u\left(\frac{p}{q}\right)$ define below. (See [13] for the original definition and basic properties of α-words.)

Let p and q be two relatively prime positive integers with $p < q$. Let $[0, a_1+1, a_2, \ldots, a_n]$ be the continued fraction expansion of $\frac{p}{q}$. Define a sequence of words $u_{-1}, u_0, u_1, \ldots, u_n$ recursively as follows: Let $u_{-1} = 1,\ u_0 = 0$, and for $1 \leq k \leq n$, let

$$u_k = \begin{cases} u_{k-2}u_{k-1}^{a_k} & (k \text{ is even}) \\ u_{k-1}^{a_k}u_{k-2} & (k \text{ is odd}). \end{cases}$$

It is know that the word u_n depends on $\frac{p}{q}$, but not the continued fraction expansion [1, 2].

Denote u_n by $u\left(\frac{p}{q}\right)$. Clearly, its first (resp., last) letter is 0 (resp., 1).

A word w is said to be an *α-word* if either $w \in \{0,1\}$ or there are two relatively prime positive integers p and q with $p < q$ such that w is a conjugate of $u\left(\frac{p}{q}\right)$. Conjugates of $u\left(\frac{F_{n-1}}{F_n}\right)$ (resp., $u\left(\frac{F_{n-2}}{F_n}\right)$) are known as *binary Fibonacci words* (see [6]).

We first report briefly some known results about the word $u = u\left(\frac{p}{q}\right)$ and its reversal. The conjugates $u, T(u), \ldots, T^{q-1}(u)$ of u are exactly the distinct α-words with length q and height p. Thus each α-word is primitive. The word u (resp., $\tilde{u}$) is a Lyndon (resp., anti-Lyndon) α-word (see [1,11]). The word u is the only binary word which has two factorizations of the form $u = xy = 0zl$, where x, y, z are palindromes, $|z| = q-2, |y| = s$ and $1 \le s < q$ is such that $ps \equiv 1 (\mathrm{mod}\ q)$ (see [20]). The conjugate class $[u]$ of u is closed under taking reversals. Clearly $\tilde{u} = T^{-s}(u)$. Both u and $\tilde{u}$ are unbordered. Furthermore, the set of Lyndon α-words and their reversals are the only unbordered finite Sturmian words (a *finite Sturmian word* is any finite factor (or segment) of any characteristic word (see section 5)) [14]. The set of Lyndon α-words coincides with the set of Christoffel primitives (see [1,2] for the definition of Christoffel primitive).

Let $[0, a_1+1, a_2, \ldots, a_n]$ be the continued fraction expansion of $\frac{p}{q}$. In [13], it was shown that a word w is a conjugate of u if and only if there are integers $r_1, \ldots, r_n$ with $0 \le r_i \le a_i, 1 \le i \le n$, and words $w_{-1}, w_0, w_1, \ldots, w_n$ such that

$$w_{-1} = 1,\ w_0 = 0,\ w_n = w,$$

$$w_i = w_{i-1}^{a_i - r_i} w_{i-2} w_{i-1}^{r_i}, \quad 1 \le i \le n.$$

In fact, each conjugate $T^k(u)$ of u corresponds to those n-tuples $(r_1, \ldots, r_n)$ of integers with $0 \le r_i \le a_i,\ 1 \le i \le n$ and $k \equiv \sum_{i=1}^{n} r_i q_{i-1} (\mathrm{mod}\ q)$, where $q_{-1} = q_0 = 1,\ q_i = a_i q_{i-1} + q_{i-2},\ 1 \le i \le n$. Thus, each α-word can be obtained recursively by concatenation. The words, having length q and height p, obtained with $r_1 = \cdots = r_n = 0$ or $r_1 = \cdots = r_{n-1} = 1 - r_n = 0$ are called standard Sturmian words (see [1]). It is not hard to see that a word w having length q and height p is a standard Sturmian word if and only if $w = T(u)$ or $w = T(\tilde{u})$.

Let $u\left(\frac{0}{1}\right) = 0$ and $u\left(\frac{1}{1}\right) = 1$. If $\frac{t}{s}$ and $\frac{t'}{s'}$ are consecutive fractions in the Farey sequence of any order with $\frac{t}{s} < \frac{t'}{s'}$, then $u\left(\frac{t+t'}{s+s'}\right) = u\left(\frac{t}{s}\right) u\left(\frac{t'}{s'}\right)$. Also the mapping $r \mapsto u(r)$ is an increasing function from the set of all reduced fractions in [0,1] onto the set of all Lyndon α-words. In other words, if $r < r'$ then $u(r) < u(r')$ in the lexicographic order (see [2]).

More results - both old and new - about $u\left(\frac{p}{q}\right)$ will be presented below.

In an earlier paper, the present author proved that if w is an α-word having length q, then $M([w])$ is a set of q consecutive positive integers and $\delta(w) = q-1$. Each of these properties actually characterizes α-words (Theorem 4.4). The result used to prove this characterization is itself a characterization of α-words (Lemma 2.1) with other interesting consequences besides Theorem 4.4. In section 3, we obtain characterization of elements of the set PER and standard Sturmian words (Corollary 3.2), and we identify those α-words that are palindromes (Corollary 3.4). In section 5, we compute the determinants of a class of matrices involving α-words (Theorem 5.1). As a special case, we obtain a sequence of (0,1)-matrices $A_1, A_2 \ldots$ such that A_n is an $F_n \times F_n$ matrix whose rows are precisely the Fibonacci words having length F_n, height F_{n-1} (resp., F_{n-2}), and $det(A_n) = F_{n-1}$ (resp., F_{n-2}).

2. A LEMMA

[11,14,16,18] present some characterizations of α-words. The characterization proved in [11] is restated in Lemma 2.1 below. With this result, we know exactly where the ones in each α-word are located and so each α-word can be generated directly without using α-words of shorter lengths. Corollary 2.2 shows how all α-words having the same length q and height p may be ordered in such a way that consecutive pairs differ in exactly two adjacent letters. Sections 3-5 present some interesting consequences of Lemma 2.1 and Corollary 2.2.

Lemma 2.1: Let p and q be relatively prime positive integers with $p < q$. Define s as the unique integer with

$$sp \equiv 1 (\text{mod } q) \text{ and } 1 \le s < q. \tag{1}$$

Let $u = u\left(\frac{p}{q}\right)$. Then for $0 \le j \le q-1$,

$$\begin{aligned} &\text{the } k^{th} \text{ letter of } T^{js}(u) \text{ is } 1 \\ \Longleftrightarrow\; & k \equiv (r-j)s \text{ (mod } q) \text{ for some } r \text{ with } 0 \le r \le p-1, \\ \Longleftrightarrow\; & k \equiv 1 + (r+j)(q-s) (\text{mod } q) \text{ for some } r \text{ with } 1 \le r \le p. \end{aligned}$$

A proof of Lemma 2.1 appears in the Appendix (see also [11]).

Corollary 2.2: Let p, q, s, and u be as in Lemma 2.1. Let $0 \le j \le q-1$. The words $T^{js}(u)$ and $T^{(j+1)s}(u)$ differ by exactly two adjacent letters. If $i \equiv (p-1-j)s$ (mod q) and $1 \le i \le q$, then the $(i-1)^{th}$ and the i^{th} letters in $T^{js}(u)$ and $T^{(j+1)s}(u)$ are 01 and 10 respectively.

Proof: Let $0 \le j \le q-1$. The positions of 1 in $T^{js}(u)$ and $T^{(j+1)s}(u)$ are respectively

$$-js, (1-j)s, \ldots, (p-2-j)s, (p-1-j)s,$$

and

$$(-j-1)s, -js, (1-j)s, \ldots, (p-2-j)s$$

(mod q). If $(p-j-1)s \equiv i \pmod q$ where $1 \le i \le q$, then clearly $i \ne 1$ and $(-j-1)s \equiv i-1 \pmod q$. Hence the words $T^{js}(u)$ and $T^{(j+1)s}(u)$ differ by exactly two letters. The $(i-1)^{th}$ and the i^{th} letters in $T^{js}(u)$ and $T^{(j+1)s}(u)$ are 01 and 10 respectively. □

We remark that when

$$q = F_n \text{ and } p = F_{n-1},\ s = \begin{cases} F_{n-1} & (n \text{ even }) \\ F_{n-2} & (n \text{ odd}) \end{cases}, n \ge 3.$$

Then Lemma 2.1 and Corollary 2.2 reduce to Theorem 2 (or Corollary 12(i) of [6]) and Theorem 3 of [10] respectively.

3. IMMEDIATE CONSEQUENCES

Throughout this section, let p, q, s, and u be as in Lemma 2.1. We shall show how Lemma 2.1 yeilds new and old results on factorization, PER, standard Sturmian words, lexicographic order, reversals and moments.

Corollary 3.1:

(a) $u = xy$, where x and y are palindromes with $|y| = s$ and $|x| = q - s$.

(b) $u = 0zl$, where z is a palindrome.

Note that, by taking reversals, we immediately derive from (a) and (b) respectively that $\tilde{u} = yx$ and $\tilde{u} = lz0$.

Proof: The proofs of (a) and (b) are almost identical so we suffice with the proof of (b). Let $2 \le k \le q-1$.

The k^{th} letter of u is 1

$\Longleftrightarrow k \equiv rs \pmod q$ for some r with $1 \le r \le p-1$ (by Lemma 2.1 with $j=0$)

$\Longleftrightarrow q+1-k \equiv (p-r)s \pmod q$ for some $1 \le r \le p-1$ (by equation (1))

$\Longleftrightarrow$ the $(q+1-k)^{th}$ letter of u is 1.

Therefore the result follows. □

Let PER= $\{0,1\} \cup \{z : 0z1$ is a Lyndon α-word$\}$. Note that the empty word belongs to PER. Let PER01= $\{z01 : z \in \text{PER}\}$. The set PER10 is defined similarly. The set of standard Sturmian words equals $\{0,1\}\cup$ PER01$\cup$PER10. Elements of PER and standard Sturmian words have been recently studied extensively (see [1]). The following corollary provides characterizations of these words.

Corollary 3.2:

(a) Let $z \in \text{PER}$ with $|z| = q-2$ and $|z|_1 = p-1 \ge 1$. Then

the k^{th} letter of z is 1

$\Longleftrightarrow k \equiv rs - 1 \pmod{q}$ for some r with $1 \leq r \leq p-1$

$\Longleftrightarrow k \equiv r(q-s) \pmod{q}$ for some r with $1 \leq r \leq p-1$.

(b) Let $w \in$ PER01 and $w' \in$ PER10 with $|w| = |w'| = q$ and $|w|_1 = |w'|_1 = p$. Then the k^{th} letter of w is 1

$\Longleftrightarrow k \equiv rs - 1 \pmod{q}$ for some r with $1 \leq r \leq p$;

the k^{th} letter of w' is 1

$\Longleftrightarrow k \equiv r(q-s) \pmod{q}$ for some r with $1 \leq r \leq p$.

Proof: Part (a) follows from Lemma 2.1 and the fact that $0z1 = u$. Part (b) follows from the fact that $w = T(\tilde{u})$ and $w' = T(u)$. □

When the conjugates of u are listed as in (2) below, we observe some interesting phenomena.

Corollary 3.3 (see [11]):

(a) The sequence of words

$$u, T^s(u), T^{2s}(u), \ldots, T^{(q-1)s}(u) = \tilde{u} \tag{2}$$

is increasing in lexicographic order.

(b) $T^{js}(u)$ have increasing moments with $M(T^{js}(u)) = \frac{(p-1)(q+1)}{2} + j + 1$ $(0 \leq j \leq q-1)$.

Proof: Part (a) and the recurrence relation $M(T^{(j+1)s}(u)) = M(T^{js}(u)) + 1, 0 \leq j \leq q-2$, follow immediately from Corollary 2.2 and the definition of M. Thus $M(T^{js}(u)) = M(u) + j, 0 \leq j \leq q-1$. We have

$$M(u) = \sum_{h=1}^{p-1}\left(q + 1 - \left(\left[\frac{hq}{p}\right] + 1\right)\right) + 1 \text{ (by definition of } M \text{ and Lemma A3 of Appendix)}$$

$$= q(p-1) - \sum_{h=1}^{p-1}\left[\frac{hq}{p}\right] + 1 \text{ (by rearrangement)}$$

$$= q(p-1) - \frac{(q-1)(p-1)}{2} + 1 \text{ (by e.g. [5])}$$

$$= \frac{(q+1)(p-1)}{2} + 1,$$

proving (b). □

The above corollary generlizes Corollaries 2 and 3 of [10]. The following corollary generalizes Lemmas 6 and 7 of [7].

Corollary 3.4:

(a) $T^{(q-1-j)s}(u) = \left(T^{js}(u)\right)^{\sim}$, $0 \le j \le q-1$.

(b) If q is odd, then $[u]$ contains exactly one palindrome, namely $T^{\left(\frac{q-1}{2}\right)s}(u)$; if q is even, $[u]$ contains no palindrome.

Note, letting $j = 0$ in (a) yields $\tilde{u} = T^{-s}(u)$.

Proof:

Let $0 \le j \le q-1$. By repeated use of Lemma 2.1, for $1 \le k \le q$,

the $(q+1-k)^{th}$ letter of $T^{(q-1-j)s}(u)$ is 1

$\iff q+1-k \equiv 1 + (r + (q-1-j))(q-s) \pmod q$ for some $1 \le r \le p$

$\iff k \equiv (r' - j)s \pmod q$ for some $0 \le r' \le p-1$

$\iff$ the k^{th} letter of $T^{js}(u)$ is 1.

This proves (a). Part (b) follows immediately from part (a) and the distinctness of the $T^j(u)$. □

4. MOMENTS OF α-WORDS

For any binary word w, let $\delta(w) = max\{M(u) - M(v) : u, v \in [w]\}$. The following lemma summarizing the properties of moments of α-words is an immediate consequence of part (b) of Corollary 3.3.

Lemma 4.1: Let w be an α-word with $|w| = q \ge 2$ and $|w|_1 = p$. Let $u = u\left(\frac{p}{q}\right)$. Then

(a) $M(u) = minM([w]) = \frac{(p-1)(q+1)}{2} + 1$, $M(\tilde{u}) = maxM([w]) = \frac{(p+1)(q+1)}{2} - 1$.

(b) $\delta(w) = q - 1$.

(c) $M([w])$ is a set of q consecutive positive integers.

We shall prove in Theorem 4.4 below that each of the conditions (b) and (c) is equivalent to saying that w is an α-word. We need the following lemma which is useful when studying moments of binary words.

Lemma 4.2: Let w be a binary word with $|w| = q$ and $|w|_1 = p$. Let $M_k = M(T^k(w))$, $0 \le k < q$. Let $w = c_1c_2\ldots c_q$ where each c_i is either 0 or 1. Define $c_{q+j} = c_j$ for $1 \le j \le q$. Then for $0 \le r < k < q$, we have

$$M_k - M_r = p(k-r) - q\sum_{i=r+1}^{k} c_i.$$

In particular, $M_k - M_0 = pk - q\sum_{i=1}^{k} c_i$ if $k > 0$.

Proof: For each k with $0 \le k \le q-1$, since $T^k(w) = c_{k+1}c_{k+2}\dots c_{k+q}$, we have

$$M_k = \sum_{j=1}^{q}(q+1-j)c_{k+j} = \sum_{i=k+1}^{k+q}(k+q+1-i)c_i = p(k+q+1) - \sum_{i=k+1}^{k+q} ic_i.$$

If $r < k$, then

$$M_k - M_r = p(k+q+1) - \sum_{j=k+1}^{k+q} jc_j - p(r+q+1) + \sum_{i=r+1}^{r+q} ic_i$$

$$= p(k-r) + \sum_{i=r+1}^{k} ic_i - \sum_{j=r+q+1}^{k+q} jc_j$$

$$= p(k-r) - q\sum_{i=r+1}^{k} c_i. \quad \square$$

Lemma 4.3: Let w be a binary word with $|w| = q \ge 2$ and $|w|_1 = p$. If $\delta(w) = q-1$ then q and p are relatively prime positive integers and w is an α-word conjugate to $u\left(\frac{p}{q}\right)$.

Proof: Let $u \in [w]$ with $M(u) = minM([w])$. Let $k_1, k_2, \dots, k_q$ be a permutation of $0, 1, \dots, q-1$ such that $k_1 = 0$ and $M_{k_1} \le M_{k_2} \le \dots \le M_{k_q}$. Let $u = c_1c_2\dots c_q$ where each c_i is either 0 or 1. Define $c_{q+j} = c_j$ for $1 \le j \le q$. By the assumption and Lemma 4.2, we have

$$q - 1 = M_{k_q} - M_{k_1} = pk_q - q\sum_{i=1}^{k_q} c_i,$$

and so q and p are relatively prime positive integers. Again by Lemma 4.2, the moments $M_{k_1}, M_{k_2}, \dots, M_{k_q}$ are all distinct and therefore $M_{k_{m+1}} - M_{k_m} = 1$, for $1 \le m \le q-1$.

Let $1 \le m \le q-1$. Lemma 4.2 also implies that

$$1 = M_{k_{m+1}} - M_{k_m} = \begin{cases} p(k_{m+1} - k_m) - q\sum_{i=k_m+1}^{k_{m+1}} c_i & (\text{if } k_m < k_{m+1}), \\ q\sum_{i=k_{m+1}+1}^{k_m} c_i - p(k_m - k_{m+1}) & (\text{if } k_{m+1} < k_m). \end{cases}$$

Define s by equation (1). Then

$$k_{m+1} - k_m = \begin{cases} s & (k_m < k_{m+1}) \\ s - q & (k_{m+1} < k_m) \end{cases}$$

$$\equiv s \pmod q$$

and therefore $k_m \equiv (m-1)s \pmod q$.

We claim that $c_{k_r} = 0$ for $p+1 \le r \le q$. To show this, let $1 \le m \le q-p$. Since $k_{m+p} - k_m \equiv (m+p-1)s - (m-1)s = ps \equiv 1 (\text{mod } q)$ and $-q+1 \le k_{m+p} - k_m \le q-1$, it follows that $k_{m+p} - k_m$ equals either $-q+1$ or 1. If $k_{m+p} - k_m = -q+1$, then $k_{m+p} = 0$ (and $k_m = q-1$). But then $m+p = 1$, a contradiction. Therefore $k_{m+p} = k_m + 1$. According to Lemma 4.2, we have

$$p = M_{k_{m+p}} - M_{k_m} = p(k_{m+p} - k_m) - q \sum_{i=k_m+1}^{k_{m+p}} c_i = p - qc_{k_{m+p}};$$

so $c_{k_{m+p}} = 0$, proving our claim.

Since $|u|_0 = q - p$, we see that

$$\begin{aligned} c_k = 1 &\iff k = q \text{ or } k_r \text{ for some } r \text{ with } 2 \le r \le p \\ &\iff k \equiv rs \pmod q \text{ for some } r \text{ with } 0 \le r \le p-1. \end{aligned}$$

It follows from Lemma 2.1 that $u = u\left(\frac{p}{q}\right)$. Consequently w is an α-word. □

Combining Lemma 4.1 and 4.3, we have the following characterization of α-words.

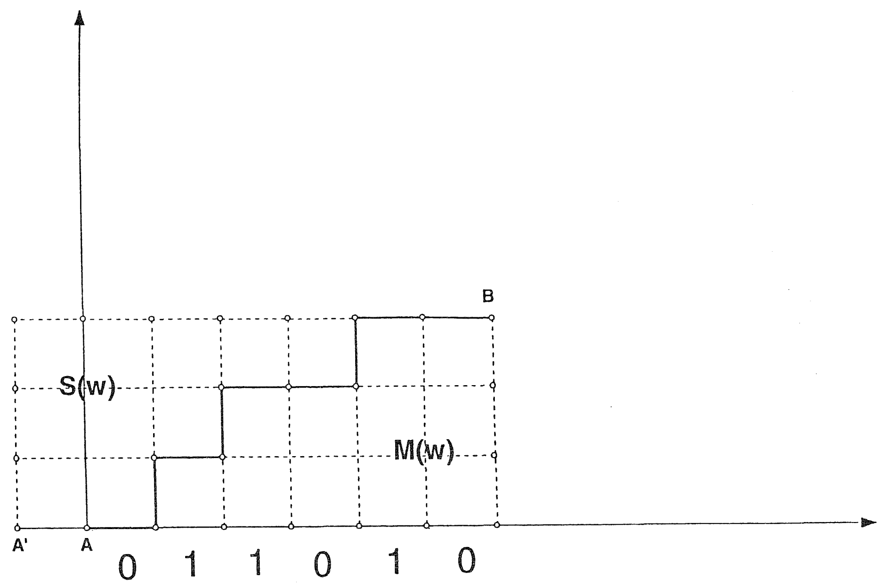
Theorem 4.4: Let w be a binary word with $|w| = q \ge 2$. Then the following statements are equivalent:

(a) $\delta(w) = q - 1$,

(b) w is an α-word,

(c) $M([w])$ is a set of q consecutive positive integers.

Remark 4.5: For $w = c_1 c_2 \dots c_q$ where each c_i is either 0 or 1, define $S(w) = \sum_{i=1}^{q} ic_i$. The results about moments can easily be reformulated using $S(w)$ instead of $M(w)$. Plainly $S(w) = M(\tilde{w})$, and $S(w) + M(w) = (|w|+1)|w|_1$. Graphically, a word w is represented by a polygonal path from $A(0,0)$ to $B(|w|, |w|_1)$ as follows: starting from the origin A, represent a 0 (resp., 1) in w by a horizontal unit segment going to the right (resp., a vertical unit segment going upward, followed by a horizontal unit segment going to the right). This polygonal path divides the rectangular region having opposite vertexes $A'(-1,0)$ and B into two subregions. The one below (resp., above) the polygonal path has area $M(w)$ (resp., $S(w)$) (see Figure).



5. DETERMINANTS OF MATRICES INVOLVING α-WORDS

Throughout this section, let q and p be relatively prime positive integers with $p < q$. Let $u = u\left(\frac{p}{q}\right)$. Regarding each binary word as a vector, we consider the $q \times q$ $(0,1)$-matrix whose j^{th} row is the α-word $T^{-(j-1)}(\tilde{u})$, $1 \le j \le q$. It is easy to see that this matrix is a circulant matrix, that is, a matrix of the form

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_{q-1} & c_q \\ c_q & c_1 & \cdots & c_{q-2} & c_{q-1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_q & c_1 \end{bmatrix}$$

where c_k is the k^{th} digit of $\tilde{u}$. We denote this matrix by $circ(\tilde{u})$ (see [19]).

Among all the matrices obtained from $circ(\tilde{u})$ by permuting its rows, the matrix $circ(\tilde{u})$ is of particular interest for the following reasons.

Let α be any irrational number between 0 and 1 such that $\frac{p}{q}$ is a convergent of the continued fraction expansion of α. The *characteristic word* $f(\alpha)$ is an infinite binary word whose k^{th} letter is $[(k+1)\alpha] - [k\alpha]$, $k \ge 1$ (see, for example, [3, 13-15, 21, 23]). When $\alpha = \frac{\sqrt{5}-1}{2}$, $f(\alpha)$ is called the *golden sequence* (see, for example, [4, 8, 9, 12, 17, 24, 25]).

Golden sequence turns out to be the Fibonacci binary word pattern $F(1, 01)$ (an infinite word $w_1w_2w_3\ldots$, where $w_1 = x$ and $w_2 = y$ are binary words, and $w_n = w_{n-2}w_{n-1}$, $n \geq 3$, is called a *Fibonacci binary word pattern* and is denoted by $F(x, y)$ (see [17, 25])).

It is well-known that for each $k \geq 1$, there are exactly $k+1$ distinct factors (or segments) of $f(\alpha)$ (see [23]). Let y denote the palindrome that differs from u only by the last (resp., first) letter if the q^{th} letter of $f(\alpha)$ is 1 (resp., 0). It was proved in [13] that for $1 \leq k \leq q$, the rows of the upper left $(k+1) \times k$ submatrix of the $(q+1) \times q$ matrix

$$\begin{bmatrix} circ(\tilde{u}) \\ y \end{bmatrix} \left(\text{resp., } \begin{bmatrix} circ(u) \\ y \end{bmatrix}\right)$$

are precisely the $k+1$ distinct factors of $f(\alpha)$ of length k.

Another interesting fact about $circ(\tilde{u})$ is contained in the following theorem.

Theorem 5.1: $\det(circ(\tilde{u})) = p$, if $q \geq 1$. Here $u\left(\frac{0}{1}\right) = 0$ and $u\left(\frac{1}{1}\right) = 1$.

Since the matrices under consideration are circulant matrices, their eigenvalues and hence their determinants can be computed using the q^{th} roots of unity. However the following row rule proof based on the combinatoric properties of Corollary 2.2 is more elegant.

Proof: Let $\tilde{u} = c_1c_2\ldots, c_q$ where $c_1, \ldots c_q \in \{0, 1\}$. Clearly the result holds for $q \leq 2$. Now let $q \geq 3$. Using (1), for $1 \leq t \leq q$, define $1 \leq i_t \leq q$ such that $i_t \equiv 1 + (t-1)s \pmod{q}$. Denote $circ(\tilde{u})$ by A and its (i, k)-entry by $A(i, k)$. For $2 \leq t \leq q$, since row i_t (resp., i_{t-1}) of A is $T^{-i_t+1}(\tilde{u}) = T^{(q-t)s}(u)$ (resp., $T^{(q-t+1)s}(u)$), Corollary 2.2 implies that

$$\begin{aligned} &A(i_{t-1}, i_t - 1) = 1,\ A(i_{t-1}, i_t) = 0, \\ &A(i_t, i_t - 1) = 0,\ A(i_t, i_t) = 1, \\ &A(i_t, k) = A(i_{t-1}, k) \text{ for } k \neq i_t \text{ and } k \neq i_t - 1. \end{aligned}$$

Let B be the matrix obtained from A by adding (-1) times row i_{t-1} to row i_t, for each $t = q, q-1, \ldots, 2$, in the order given. Then

$$\begin{aligned} B(1, k) &= A(1, k) = c_k, \\ B(i_t, k) &= (-1)A(i_{t-1}, k) + A(i_t, k) \\ &= \begin{cases} -1 \ (k = i_t - 1) \\ 1 \ (k = i_t) \\ 0 \ (\text{otherwise}), \end{cases} \end{aligned}$$

where $2 \leq t \leq q$, and $1 \leq k \leq q$. Since $i_2, i_3, \ldots, i_q$ is a permutation of $2, 3, \ldots, q$, it follows that B is the matrix

$$\begin{bmatrix} c_1 & c_2 & c_3 & \ldots & c_{q-1} & c_q \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{bmatrix}$$

Clearly,

$$det(circ(\tilde{u})) = det(B) = \sum_{k=1}^{q} c_k = p. \quad \square$$

Here is a special case of Theorem 5.1. Let $\{v_n\}$ and $\{z_n\}$ be sequences of Fibonacci words given recursively by

$$v_0 = 1, v_1 = 0, v_2 = 1, v_n = \begin{cases} v_{n-1}v_{n-2} & (n \text{ is odd}) \\ v_{n-2}v_{n-1} & (n \text{ is even}), \end{cases}$$

$$z_1 = 1, z_2 = 0, z_n = \begin{cases} z_{n-2}z_{n-1} & (n \text{ is odd}) \\ z_{n-1}z_{n-2} & (n \text{ is even}), \end{cases}$$

Let $A_n = circ(v_n)$ (resp., $circ(z_n)$), $n \geq 1$. Since $\frac{F_{n-1}}{F_n} = [0, 1, 1, \ldots, 1]$ ($n - 1$ ones) (resp., $\frac{F_{n-2}}{F_n} = [0, 2, 1, \ldots, 1]$ ($n - 3$ ones)), $n \geq 3$, we see that $v_n = \left(u\left(\frac{F_{n-1}}{F_n}\right)\right)^{\sim}$ $\left(\text{resp., } z_n = \left(u\left(\frac{F_{n-2}}{F_n}\right)\right)^{\sim}\right)$, $n \geq 1$. It follows from Theorem 5.1 that each A_n is an $F_n \times F_n$ $(0,1)$ - matrix whose rows are precisely the Fibonacci words having length F_n and height F_{n-1} (resp., F_{n-2}) and $det(A_n) = F_{n-1}$ (resp., F_{n-2}).

APPENDIX. A PROOF OF LEMMA 2.1

For each real number θ, the infinite binary word $f(\theta)$ whose k^{th} letter is $[(k+1)\theta] - [k\theta]$, $k \geq 1$, is called the *characteristic word* of θ.

Lemma A1 (see [21]): Let $0 < \theta < 1$.

(a) If θ is irrational and $k \geq 1$, then

$$\text{the } k^{th} \text{ letter of } f(\theta) \text{ is } 1$$

$$\Longleftrightarrow k = \left[\frac{h}{\theta}\right] \text{ for some } h \geq 1.$$

(b) If $\theta = \frac{p}{q}$ is rational, where p, q are relatively prime positive integers, and $k \geq 1, k \not\equiv 0$ and $k \not\equiv -1 \pmod{q}$, then

$$\text{the } k^{th} \text{ letter of } f(\theta) \text{ is } 1$$

$$\Longleftrightarrow k = \left[\frac{h}{\theta}\right] \text{ for some } h \geq 1,\ h \not\equiv 0 \pmod{p}.$$

Throughout the rest of this section, let p and q be relatively prime positive integers with $p < q$. Let $1 \leq s < q, 1 \leq t < p$, and $ps = qt + 1$. Let $u = u\left(\frac{p}{q}\right)$. If w is a word and $w = xy$ where y is nonempty, we write $x = wy^{-1}$.

Lemma A2: Let θ be a real number between 0 and 1 such that $\frac{p}{q}$ is a convergent of the continued fraction expansion of θ. Let z be a palindrome such that $u = 0z1$.

(a) (see [1,3,21]) z is a prefix of $f(\theta)$.

(b) If $\frac{p}{q} > \theta$, then $u1^{-1}$ (resp., $\tilde{u}$) is a prefix of $0f(\theta)$ (resp., $1f(\theta)$), but u is not a prefix of $0f(\theta)$.

(c) If $\frac{p}{q} \leq \theta$, then u (resp., $\tilde{u}0^{-1}$) is a prefix of $0f(\theta)$ (resp., $1f(\theta)$), but $\tilde{u}$ is not a prefix of $1f(\theta)$.

(d) $0f\left(\frac{p}{q}\right) = u^{\infty}$.

Proof: Part (b) and (c) follow from (a) and the fact that $[(q-1)\theta] = p-1, [(q+1)\theta] = p$, and

$$[q\theta] = \begin{cases} p-1 & \left(\frac{p}{q} > \theta\right) \\ p & \left(\frac{p}{q} \leq \theta\right). \end{cases}$$

Part (d) follows from (b). □

The following lemma follows from Lemmas A1 and A2.

Lemma A3: The first (resp., last) letter of u is 0 (resp., 1). For $1 < k < q$,

$$\text{the } k^{th} \text{ letter of } u \text{ is } 1$$

$$\Longleftrightarrow k-1 = \left[\frac{hq}{p}\right] \text{ for some } 1 \le h \le p-1.$$

Lemma A4: For each h with $1 \le h \le p$, there is a unique r with $1 \le r \le p$ such that $\left[\frac{hq}{p}\right] \equiv rs - 1 \pmod q$. The mapping $h \longmapsto r$ is a bijection from $\{1, 2, \dots, p\}$ onto itself. Furthermore,

(a) $h \equiv rt$ and $r \equiv h(p-m) \pmod p$, where $1 \le m \le p$, and $q \equiv m \pmod p$.

(b) $h = p \Longleftrightarrow r = p$.

Proof: Let $1 \le h \le p$. Since s and q are relatively prime, there is a unique integer r, $1 \le r \le q$ such that

$$\left[\frac{hq}{p}\right] \equiv rs - 1 \pmod q.$$

Clearly (b) holds. Let n be an integer such that $\left[\frac{hq}{p}\right] = rs - 1 - nq$. Then

$$p\left[\frac{hq}{p}\right] = rps - p - nqp$$

$$= r(qt+1) - p - nqp$$
$$= q(rt - np) + r - p.$$

Since $p\left[\frac{hq}{p}\right] \le hq < p\left[\frac{hq}{p}\right] + p$, we have

$$(rt - np) + \frac{r}{q} - \frac{p}{q} \le h < rt - np + \frac{r}{q},$$

that is,

$$h + np - rt < \frac{r}{q} \le h + np - rt + \frac{p}{q}.$$

Therefore $h + np - rt = \left[\frac{r}{q}\right] = 0$ and $r - p \le q(h + np - rt) = 0$; so $h \equiv rt \pmod p$ and $1 \le r \le p$. The second part of (a) follows immediately from the first part.

It remains to show that if $1 \le h_1 < h_2 \le p$, then $\left[\frac{h_1 q}{p}\right] \not\equiv \left[\frac{h_2 q}{p}\right] \pmod q$. Let $k = h_2 - h_1$, where $1 \le h_1 < h_2 \le p$, i.e., $1 \le k \le p-1$. Then

$$\left[\frac{h_1 q}{p}\right] + 1 < \left[\frac{h_1 q}{p}\right] + k\frac{q}{p} \le \frac{h_1 q}{p} + \frac{kq}{p} = \frac{h_2 q}{p}$$

$$\le \frac{h_1 q}{p} + \frac{p-1}{p} q < \frac{h_1 q}{p} + q - 1$$

$$< \left[\frac{h_1 q}{p}\right] + q;$$

so the result follows. □

Lemma 2.1 now follows immediately from Lemmas A3 and A4.

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