AN ELEMENTARY PROOF OF JACOBI'S FOUR-SQUARE THEOREM

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1. INTRODUCTION

Recall that $\mathbb{P} := \{1,2,3,\ldots\}, \mathbb{N} := \mathbb{P} \cup \{0\}$ and $\mathbb{Z} := \{0 \pm 1, \pm 2, \ldots\}$. Then, for each $n \in \mathbb{N}$,

$$r_4(n) := |\{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 | n = m_1^2 + m_2^2 + m_3^2 + m_4^2\}|.$$

For each $n \in \mathbb{P}, \sigma(n)$ denotes the sum of all positive divisors of n, $b(n)$ denotes the exponent of the largest power of 2 dividing n, and then $Od(n) := n2^{-b(n)}$. (Quite properly, $b(n)$ (or $2^{b(n)}$) is called the binary part of n and $Od(n)$ is called the odd part of n.) In this note we give a simple proof of the following elegant result first stated and proved by Jacobi [1, p. 285].

Theorem 1: *For each* $n \in \mathbb{P}$,

$$r_4(n) = 8(2 + (-1)^n)\sigma(Od(n)).$$

(Of course, $r_4(0) = 1$.*)*

Our proof depends on several immediate consequences of the celebrated Gauss-Jacobi triple-product identity

$$\prod_{n=1}^{\infty}(1 - x^{2n})(1 + tx^{2n-1})(1 + t^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2} t^n, \tag{1}$$

which is valid for each pair of complex numbers t, x such that $t \neq 0$ and $|x| < 1$. For a proof see [2, pp. 282-283].

2. PROOF OF THEOREM 1

We begin with Jacobi's triangular-number identity [2, p. 285]

$$2\prod_{n=1}^{\infty}(1 - x^n)^3 = \sum_{-\infty}^{\infty}(-1)^k(2k+1)x^{k(k+1)/2}, \tag{2}$$

valid for each x such that $|x| < 1$. In (2) we first let $x \to x^8$, and then multiply the resulting identity by x to get

$$2x\prod_{1}^{\infty}(1 - x^{8n})^3 = \sum_{-\infty}^{\infty}(-1)^k(2k+1)x^{(2k+1)^2}. \tag{3}$$

Next, we square both sides of (3), and appeal to the elementary identity

$$u^2+v^2=\frac{1}{2}\{(u+v)^2+(u-v)^2\}$$

to get

$$4x^2\prod_1^\infty(1-x^{8n})^6=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^{j+k}(2j+1)(2k+1)x^{(2j+1)^2+(2k+1)^2}$$

$$=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^{j+k}(2j+1)(2k+1)x^{2[(j+k+1)^2+(j-k)^2]}.$$

Now, with

$$E:=\{(r,s)\in\mathbb{Z}^2|r \text{ and } s \text{ have the same parity}\},$$

it follows easily that the function $F:\mathbb{Z}^2\to\mathbb{Z}^2$, defined by

$$F(j,k):=(j+k,j-k),\ \text{for each } (j,k)\in\mathbb{Z}^2,$$

is one-to-one from $\mathbb{Z}^2$ onto E. Hence, in the foregoing identity let $r=j+k, s=j-k$, so that $j=(1/2)(r+s),\ k=(1/2)(r-s)$, and let $x\to x^{1/2}$ to get

$$4x\prod_1^\infty(1-x^{4n})^6=\sum_{(r,s)\in E}(-1)^r(r+1+s)(r+1-s)x^{(r+1)^2+s^2}$$

$$=\sum_{(r,s)\in E}(-1)^r\{(r+1)^2-s^2\}x^{(r+1)^2+s^2}$$

$$=\sum_{-\infty}^{\infty}(2m+1)^2x^{(2m+1)^2}\sum_{-\infty}^{\infty}x^{(2n)^2}-\sum_{-\infty}^{\infty}x^{(2m+1)^2}\sum_{-\infty}^{\infty}(2n)^2x^{(2n)^2}$$

$$-\sum_{-\infty}^{\infty}(2m+2)^2x^{(2m+2)^2}\sum_{-\infty}^{\infty}x^{(2n+1)^2}+\sum_{-\infty}^{\infty}x^{(2m+2)^2}\sum_{-\infty}^{\infty}(2n+1)^2x^{(2n+1)^2}$$

$$=2\left\{\sum_{-\infty}^{\infty}(2m+1)^2x^{(2m+1)^2}\sum_{-\infty}^{\infty}x^{(2n)^2}-\sum_{-\infty}^{\infty}x^{(2m+1)^2}\sum_{-\infty}^{\infty}(2n)^2x^{(2n)^2}\right\},$$

since $m \in \mathbb{Z} \Longleftrightarrow m+1 \in \mathbb{Z}$. We cancel a factor of 2 and put

$$f(x) := \sum_{-\infty}^{\infty} x^{(2m+1)^2}, \; g(x) := \sum_{-\infty}^{\infty} x^{(2n)^2}$$

to get

$$2x \prod_{1}^{\infty} (1-x^{4n})^6 = g(x)^2 \frac{\theta_x f(x) \cdot g(x) - f(x) \cdot \theta_x g(x)}{g(x)^2} \tag{4}$$

where $\theta_x := xD_x, D_x$ denoting differentiation with respect to x. But, with the help of (1), we get

$$f(x) = 2x \prod_{1}^{\infty} (1-x^{8n})(1+x^{8n})^2,$$

$$g(x) = \prod_{1}^{\infty} (1-x^{8n})(1+x^{8n-4})^2,$$

so that

$$\frac{f(x)}{g(x)} = 2x \prod_{1}^{\infty} \frac{(1+x^{8n})^2}{(1+x^{8n-4})^2}.$$

Hence,

$$\theta_x\{f(x)/g(x)\} = \frac{f(x)}{g(x)} \left\{ 1 + 16 \sum_{k=1}^{\infty} \frac{kx^{8k}}{1+x^{8k}} - 8 \sum_{k=1}^{\infty} \frac{(2k-1)x^{8k-4}}{1+x^{8k-4}} \right\}.$$

Now,

$$g(x)^2 \frac{f(x)}{g(x)} = f(x)g(x) = 2x \prod_{1}^{\infty} (1-x^{8n})^2 (1+x^{4n})^2.$$

With the help of Euler's identity [2, p. 277]

$$\prod_1^\infty (1+x^n)(1-x^{2n-1}) = 1,$$

which is valid for each complex number x such that $|x| < 1$, we substitute the foregoing evaluations into (4), cancel $2x$, let $x \longrightarrow x^{1/4}$ and divide both sides of the resulting identity by $\prod(1-x^{2n})^2(1+x^n)^2$ to get

$$\prod_1^\infty \frac{(1-x^{2n})^6(1-x^{2n-1})^6}{(1-x^{2n})^2(1+x^n)^2} = \prod_1^\infty (1-x^{2n})^4(1-x^{2n-1})^8$$

$$= 1 + 16\sum_{k=1}^\infty \frac{kx^{2k}}{1+x^{2k}} - 8\sum_{k=1}^\infty \frac{(2k-1)x^{2k-1}}{1+x^{2k-1}}. \tag{5}$$

We now digress momentarily to make a couple of key observations. First, we let $t = 1$ in (1), and observe that the fourth power of the right-hand side of the resulting identity generates the sequence $r_4(n)$, $n \in \mathbb{N}$. In other words,

$$\prod_1^\infty (1-x^{2n})^4(1+x^{2n-1})^8 = \left\{\sum_{-\infty}^\infty x^{n^2}\right\}^4 = \sum_{n=0}^\infty r_4(n)x^n.$$

Next, we observe that the composite function $\sigma \circ Od$ arises quite naturally in the expansion:

$$\sum_{k=1}^\infty \frac{(2k-1)x^{2k-1}}{1-x^{2k-1}} = \sum_{k=1}^\infty\sum_{j=0}^\infty (2k-1)x^{2k-1}\cdot x^{j(2k-1)}$$

$$= \sum_{k=1}^\infty\sum_{j=1}^\infty (2k-1)x^{j(2k-1)}$$

$$= \sum_{n=1}^\infty x^n \sum_{\substack{d|n \\ d|\text{odd}}} d$$

$$= \sum_{n=1}^\infty \sigma(Od(n))x^n.$$

Returning to the proof of our theorem, we appeal to [2, p. 312], and in (5) let $x \to -x$ to get

$$\sum_{n=0}^{\infty} r_4(n)x^n = \prod_{1}^{\infty}(1-x^{2n})^4(1+x^{2n-1})^8$$

$$= 1 + 16\sum_{k=1}^{\infty} \frac{kx^{2k}}{1+x^{2k}} + 8\sum_{k=1}^{\infty} \frac{(2k-1)x^{2k-1}}{1-x^{2k-1}}$$

$$= 1 + 16\sum_{n=1}^{\infty} \frac{(2n-1)x^{4n-2}}{1-x^{4n-2}} + 8\sum_{k=1}^{\infty} \frac{(2k-1)x^{2k-1}}{1-x^{2k-1}}$$

$$= 1 + 16\sum_{n=1}^{\infty} \sigma(Od(n))x^{2n} + 8\sum_{n=1}^{\infty} \sigma(Od(n))x^n$$

$$= 1 + 16\sum_{n=1}^{\infty} \sigma(Od(2n))x^{2n} + 8\sum_{n=1}^{\infty} \sigma(Od(2n))x^{2n} + 8\sum_{n=1}^{\infty} \sigma(2n-1)x^{2n-1}$$

$$= 1 + 24\sum_{n=1}^{\infty} \sigma(Od(2n))x^{2n} + 8\sum_{n=1}^{\infty} \sigma(2n-1)x^{2n-1}.$$

Here, we've made use of the obvious facts: $Od(2n) = Od(n)$ and $Od(2n-1) = 2n-1$, for each $n \in \mathbb{P}$. Finally, we equate coefficients of like powers of x to get

$$r_4(0) = 1$$

and for each $n \in \mathbb{P}$,

$$r_4(2n) = 24\sigma(Od(2n)),\ r_4(2n-1) = 8\sigma(2n-1).$$

This completes the proof of theorem 1.

REFERENCES

[1] L.E. Dickson. *History of the Theory of Numbers.* Volume II. New York: Chelsea, 1952.

[2] G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers.* Clarendon Press, Oxford (1960), 4th ed.

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