ON SOME CLASSES OF EFFECTIVELY INTEGRABLE DIFFERENTIAL EQUATIONS AND FUNCTIONAL RECURRENCES

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1. INTRODUCTION

It is well-known (see, [4] p. 411) that the general solution of the differential equation $(x^2-1)y''+xy'-n^2y=0$ is of the form:

$$y=C_1\left(\frac{x+\sqrt{x^2-1}}{2}\right)^n+C_2\left(\frac{x-\sqrt{x^2-1}}{2}\right)^n, \tag{1}$$

where C_1 and C_2 are arbitrary constants and $n\in N$.

For $C_1=C_2=1$ from (1) we get that

$$T_n(x)=\left(\frac{x+\sqrt{x^2-1}}{2}\right)^n+\left(\frac{x-\sqrt{x^2-1}}{2}\right)^n, \tag{2}$$

is the Chebyshev polynomial of the first kind.

In [2] the author has considered a more general class of polynomials, namely:

$$W_n(x;c)=\left(\frac{x+\sqrt{x^2+c}}{2}\right)^n+\left(\frac{x-\sqrt{x^2+c}}{2}\right)^n, \tag{3}$$

where c is a parameter and where $n\geq 1$ is the degree of the polynomial $W_n(x;c)$. Moreover, it has been proved in [2] that the function:

$$y=C_1\left(\frac{x+\sqrt{x^2+c}}{2}\right)^n+C_2\left(\frac{x-\sqrt{x^2+c}}{2}\right)^n, \tag{4}$$

is the general solution of the differential equation:

$$(x^2+c)y''+xy'-n^2y=0, x^2+c>0,\ n\in N. \tag{*}$$

The polynomial $W_n(x;c)$ given by (3) contains the well-known Pell polynomial when $c = 1$ and the Fibonacci polynomial when $c = 4$.

In this paper we give further extensions of this result.

2. BASIC LEMMAS

Lemma 1: Let s_0, $u \in C^2(J)$ be real-valued functions of x, where $J = (x_1, x_2) \subset R$ and $u \neq 0$ on J. The function $y_1 = s_0 u^\lambda$, with non-zero real constant λ, is the particular solution of the differential equation:

$$D_0 y'' + D_1 y' + D_2 y = 0 \tag{2.1}$$

if and only if there exist the functions $s_1, s_2 \in C^2(J)$ such that

$$D_0 s_2 + D_1 s_1 + D_2 s_0 = 0. \tag{2.2}$$

Proof: Suppose that the function $y_1 = s_0 u^\lambda$ is the particular solution of (2.1). Then we have $D_0 y_1'' + D_1 y_1' + D_2 y_1 = 0$ and by the assumption on the functions s_0 and u it follows that

$$y_1' = s_0' u^\lambda + s_0 \lambda u^{\lambda-1} u' = u^\lambda \left(s_0' + \lambda s_0 \frac{u'}{u} \right). \tag{2.3}$$

Putting

$$s_1 = s_0' + \lambda s_0 \frac{u'}{u} \tag{2.4}$$

in (2.3) we have $y_1' = s_1 u^\lambda$. In a similar manner we obtain

$$y_1'' = \left(s_1 u^\lambda\right)' = s_1' u^\lambda + \lambda s_1 u^{\lambda-1} u' = u^\lambda \left(s_1' + \lambda s_1 \frac{u'}{u} \right). \tag{2.5}$$

Putting

$$s_2 = s_1' + \lambda s_1 \frac{u'}{u} \tag{2.6}$$

in (2.5) we have $y_1'' = s_2 u^\lambda$, and therefore we obtain $D_0 y_1'' + D_1 y_1' + D_2 y_1 = D_0 s_2 u^\lambda + D_1 s_1 u^\lambda + D_2 s_0 u^\lambda = u^\lambda (D_0 s_2 + D_1 s_1 + D_2 s_0) = 0$.

Since $u \neq 0$ on J then (2.2) follows from the last equality. Now, we suppose that (2.2) is satisfied by some functions $s_0, s_1, s_2 \in C^2(J)$. Then we have

$$D_0 s_2 u^\lambda + D_1 s_1 u^\lambda + D_2 s_0 u^\lambda = 0. \tag{2.7}$$

Putting $y_1 = s_0 u^\lambda$ in (2.7) we obtain $y_1' = s_1 u^\lambda$ and $y_1'' = s_2 u^\lambda$, where the functions s_1 and s_2 are defined by the formulas (2.4) and (2.6), respectively. Hence, $D_0 y_1'' + D_1 y_1' + D_2 y_1 = 0$, and the proof of Lemma 1 is complete. □

Lemma 2: Let $s_0, t_0, u, v \in C^2(J)$ be real-valued functions of x and let $u \neq 0,\ v \neq 0$ on J. Then the functions

$$y_1 = s_0 u^\lambda \quad \text{and} \quad y_2 = t_0 v^\lambda \tag{2.8}$$

are particular solutions of the differential equation:

$$D_0 y'' + D_1 y' + D_2 y = 0, \tag{2.9}$$

if and only if the functions s_1, t_1, s_2, and t_2 are given by the formulas:

$$s_1 = s_0' + \lambda s_0 \frac{u'}{u}, t_1 = t_0' + \lambda t_0 \frac{v'}{v}, s_2 = s_1' + \lambda s_1 \frac{u'}{u}, t_2 = t_1' + \lambda t_1 \frac{v'}{v}, \tag{2.10}$$

and

$$D_0 = \det \begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}, D_1 = \det \begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix}, D_2 = \det \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}. \tag{2.11}$$

Proof: From Lemma 1 it follows that the functions $y_1 = s_0 u^\lambda$ and $y_2 = t_0 v^\lambda$ are particular solutions of the equation (2.9) if and only if

$$D_0 s_2 + D_1 s_1 + D_2 s_0 = 0 \text{ and } D_0 t_2 + D_1 t_1 + D_2 t_0 = 0, \tag{2.12}$$

where the functions s_1, s_2, t_1, and t_2 are defined by the formulas in (2.10). Now, we consider the determinant:

$$W_1 = \det \begin{pmatrix} s_0 & s_1 & s_2 \\ s_0 & s_1 & s_2 \\ t_0 & t_1 & t_2 \end{pmatrix}. \tag{2.13}$$

It is easy to see that $W_1 = 0$, and by Laplace's theorem we obtain

$$s_0 \det \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} + s_1 \det \begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix} + s_2 \det \begin{pmatrix} s_0 & s_2 \\ t_0 & t_1 \end{pmatrix} = 0. \tag{2.14}$$

Denoting $D_0 = \det\begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}, D_1 = \det\begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix}, D_2 = \det\begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}$, in (2.14) we obtain $D_0 s_2 + D_1 s_1 + D_2 s_0 = 0$. In a similar manner we consider the determinant:

$$W_2 = \det\begin{pmatrix} t_0 & t_1 & t_2 \\ t_0 & t_1 & t_2 \\ s_0 & s_1 & s_2 \end{pmatrix}. \tag{2.15}$$

As in the previous case we obtain that $D_0 t_2 + D_1 t_1 + D_2 t_0 = 0$ and the proof of Lemma 2 is complete. □

From Lemma 1 and Lemma 2 we deduce the following lemma:

Lemma 3: Let λ be a non-zero real constant and let $u, v \in C^2(J)$ be a non-zero real-valued functions, linearly independent over R, where $J = (x_1, x_2) \subset R$. Then the general solution of the differential equation:

$$\det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} y'' + \det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} y' + \lambda \det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} y = 0, \tag{**}$$

where $g = \frac{u''}{u} - (1-\lambda)\left(\frac{u'}{u}\right)^2$ and $h = \frac{v''}{v} - (1-\lambda)\left(\frac{v'}{v}\right)^2$ is of the form

$$y = C_1 u^\lambda + C_2 v^\lambda, \tag{2.16}$$

where C_1 and C_2 are arbitrary constants.

Proof: Putting $s_0 = t_0 = 1$ in Lemma 1 and Lemma 2, we obtain $s_1 = \lambda\frac{u'}{u}, t_1 = \lambda\frac{v'}{v}$ and $s_2 = s_1' + \lambda s_1 \frac{u'}{u} = \lambda\left(\frac{u''}{u} - (1-\lambda)\left(\frac{u'}{u}\right)^2\right) = \lambda g$, $t_2 = t_1' + \lambda t_1 \frac{v'}{v} = \lambda\left(\frac{v''}{v} - (1-\lambda)\left(\frac{v'}{v}\right)^2\right) = \lambda h$. Hence, we have

$$D_0 = \det\begin{pmatrix} 1 & \lambda\frac{u'}{u} \\ 1 & \lambda\frac{v'}{v} \end{pmatrix} = \lambda \det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} \tag{2.17}$$

$$D_1 = \det\begin{pmatrix} s_2 & 1 \\ t_2 & 1 \end{pmatrix} = \det\begin{pmatrix} \lambda g & 1 \\ \lambda h & 1 \end{pmatrix} = \lambda \det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} \tag{2.18}$$

$$D_2 = \det\begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} = \det\begin{pmatrix} \lambda\frac{u'}{u} & \lambda g \\ \lambda\frac{v'}{v} & \lambda h \end{pmatrix} = \lambda^2 \det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix}. \tag{2.19}$$

From (2.17)-(2.19) it follows that equation (2.9) reduces to (**), hence by Lemma 2 it follows that the functions $y_1 = u^\lambda$, and $y_2 = v^\lambda$ are particular solutions of (**). It suffices to prove that the functions y_1 and y_2 are linearly independent over R. To this end consider the Wronskian of these functions

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} u^\lambda & v^\lambda \\ \lambda u^{\lambda-1}u' & \lambda v^{\lambda-1}v' \end{pmatrix} = \lambda u^{\lambda-1}v^{\lambda-1}\det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}. \tag{2.20}$$

By the assumptions that $u \neq 0$, $v \neq 0$ it follows that $\det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \neq 0$ on J and consequently from (2.20) we see that $W(y_1, y_2) \neq 0$ on J. Therefore the function

$$y = C_1 y_1 + C_2 y_2 = C_1 u^\lambda + C_2 v^\lambda$$

is the general solution of the differential equation (**). The proof of Lemma 3 is complete. □

3. THE RESULTS

In this part of our paper we obtain some new classes of second order differential equations which are effectively integrable and with general solutions given in explicit form (Cf. [4]). Namely, we prove of the following theorem.

Theorem 1: Let the functions $a, b \in C^2(J)$, $J = (x_1, x_2) \subset R$ be real-valued and non-zero in x such that $ax \neq \pm bx$ on J, and let a, b be linearly independent over R. Then the function

$$y = C_1(a(x) + b(x))^n + C_2(a(x) - b(x))^n \tag{3.1}$$

where C_1 and C_2 are arbitrary constants and $n \in N$ is a general solution of the differential equation:

$$P_0(x)y'' + P_1(x)y' + nP_2(x)y = 0, \tag{***}$$

where

$$P_0(x) = (a(x)^2 - b(x)^2)(a'(x)b(x) - b'(x)a(x)) = F(x)G(x) \tag{3.2}$$

$$P_1(x) = (a''(x)b(x) - b''(x)a(x))F(x) + 2(n-1)G(x)(a'(x)a(x) - b'(x)b(x)) \tag{3.3}$$

$$P_2(x) = (b''(x)a'(x) - a''(x)b'(x))F(x) - (n-1)\left((a'(x))^2 - (b'(x))^2\right)G(x) \tag{3.4}$$

Proof: Let $u = a(x) - b(x),\ v = a(x) + b(x)$ and let $y_1 = u^n$ and $y_2 = v^n$, where $n \in N$. Then by Lemma 3 it follows that

$$\det\begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} = -2\frac{a'(x)b(x) - b'(x)a(x)}{a(x)^2 - b(x)^2} = -2\frac{G(x)}{F(x)} \tag{3.5}$$

$$\det\begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} = \frac{2(a''(x)b(x) - b''(x)a(x))}{F(x)} + 4(n-1) = \frac{a'(x)a(x) - b'(x)b(x)}{F^2(x)}G(x) \tag{3.6}$$

$$\det\begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} = \frac{2(b''(x)a(x)' - a''(x)b'(x))}{F(x)} - 2(n-1) = \frac{((a'(x))^2 - (b'(x))^2)}{F(x)}G(x). \tag{3.7}$$

Substituting (3.5)-(3.7) in (**) of Lemma 3 we obtain, after some calculation, that (**) reduces to the equation $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ with the functional coefficients $P_0(x), P_1(x)$, and $P_2(x)$ as given by the formulas (3.2)-(3.4). It remains to prove that the functions $u = a(x) - b(x)$ and $v = a(x) + b(x)$ are linearly independent over R under the assumption that the functions $a(x)$ and $b(x)$ are linearly independent over R. To this end we consider the Wronskian

$$W(u,v) = \det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} = \det\begin{pmatrix} a(x) - b(x) & a(x) + b(x) \\ a'(x) - b'(x) & a'(x) + b'(x) \end{pmatrix}.$$

From the well-known properties of determinants it follows that

$$W(u,v) = 2\det\begin{pmatrix} a(x) & b(x) \\ a'(x) & b'(x) \end{pmatrix}. \tag{3.8}$$

From (3.8) and by the assumptions of the theorem about the functions a and b it folllows that $W(u,v) \neq 0$ on J and the proof of Theorem 1 is complete. □

Using Theorem 1 we obtain the following:

Theorem 2: The general solution of the differential equation

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \tag{I}$$

with coefficients $F_0(x), F_1(x)$, and $F_2(x)$ given by the formulas

$$F_0(x) = 2(bx+c)(bx+2c)(x^2+bx+c) \tag{II}$$
$$F_1(x) = \Delta x(bx+c) + 2(n-1)b(bx+2c)(x^2+bx+c)$$

$$F_2(x) = \frac{1}{2}(2\Delta(bx+c) + \Delta(n-1)(bx+2c))$$

where $\Delta = b^2 - 4c$ is the discriminant of the polynomial $f(x) = x^2 + bx + c$ and $bx + c \neq 0$ and $bx + 2c \neq 0$ on $J = (x_1, x_2) \subset R$ is of the form

$$y = C_1 \left(\frac{x + \sqrt{x^2+bx+c}}{2} \right)^n + C_2 \left(\frac{x - \sqrt{x^2+bx+c}}{2} \right)^n, \tag{III}$$

where C_1 and C_2 are arbitrary constants and $n \in N$.

Proof: Let $a(x) = \frac{x}{2}$ and $b(x) = \frac{1}{2}\sqrt{x^2+bx+c}$. Then we have $a'(x) = \frac{1}{2}$ and

$$b'(x) = \frac{2x+b}{4\sqrt{x^2+bx+c}}, \text{ so } a''(x) = 0 \text{ and } b''(x) = -\frac{\Delta}{8(x^2+bx+c)\sqrt{x^2+bx+c}}.$$

Using formulas (3.2)-(3.4) from Theorem 1 we obtain

$$P_0(x) = -\frac{(bx+c)(bx+2c)}{32\sqrt{x^2+bx+c}},$$

$$P_1(x) = -\frac{\Delta x(bx+c) + 2(n-1)b(bx+2c)(x^2+bx+c)}{64(x^2+bx+c)\sqrt{x^2+bx+c}},$$

$$P_2(x) = \frac{2\Delta(bx+c) + \Delta(n-1)(bx+2c)}{128(x^2+bx+c)\sqrt{x^2+bx+c}}.$$

From the last formulas it is easy to see that the equation reduces to the equation (I) with the coefficients given by (II). Therefore, it remains to prove that the functions $a(x) = \frac{x}{2}$ and

$b(x)=\frac{1}{2}\sqrt{x^2+bx+c}$ are linearly independent over R, if $bx+2c\neq 0$ on J. Let $W(a,b)$ denotes the Wronskian of the functions a and b. Then we have

$$W(a,b)=\det\begin{pmatrix} a(x) & b(x) \\ a'(x) & b'(x)\end{pmatrix}=\det\begin{pmatrix} \frac{x}{2} & \frac{1}{2}\sqrt{x^2+bx+c} \\ \frac{1}{2} & \frac{2x+b}{4\sqrt{x^2+bx+c}}\end{pmatrix}=-\frac{bx+2c}{8\sqrt{x^2+bx+c}}.$$

From the last equality it follows that $W(a,b)\neq 0$ on J, because $bx+2c\neq 0$ on J.

The proof of Theorem 2 is complete. □

Now, we observe that the result described in Introduction follows immediately from Theorem 2 in the particular case where $b=0$.

4. FUNCTIONAL RECURRENCES AND GENERALIZED HORADAM-MAHON FORMULA FOR PELL POLYNOMIALS

In [3], Horadam and Mahon consider a matrix method in the investigation of some classes of polynomials such as the Pell polynomials $P_n(x)$. They proved that for every natural number n, we have

$$P_{n-1}(x)P_{n+1}(x)-P_n^2(x)=(-1)^n, \tag{4.1}$$

where $P_n(x)$ is defined by the recurrence formula:

$$P_0(x)=0,\ P_1(x)=1,\ P_{n+2}(x)=2xP_{n+1}(x)+P_n(x). \tag{4.2}$$

In [1], the authors have considered the functional matrix

$$A=A(x)=\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x)\end{pmatrix}.$$

Let $TrA(x)\neq 0$ or $\det A(x)\neq 0$ on $J=(x_1,x_2)\subset R$ and let

$$r=r(x)=TrA(x)=a(x)+d(x), s=s(x)=-\det A(x), \tag{4.3}$$

and

$$u_0=u_0(x)=r,\quad u_1=u_1(x)=ru_0(x)+s. \tag{4.4}$$

Let

$$u_n(x)=ru_{n-1}(x)+su_{n-2}(x),\quad \text{for } n\geq 2, \tag{4.5}$$

be a functional recurrence sequence associated with the matrix $A = A(x)$. Then for every natural number $n \geq 2$, we have, in [1],

$$A^n(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}^n = \begin{pmatrix} a(x)u_{n-2}(x) + v_{n-2}(x) & b(x)u_{n-2}(x) \\ c(x)u_{n-2}(x) & d(x)u_{n-2}(x) + v_{n-2}(x) \end{pmatrix}, \tag{4.6}$$

where

$$v_{n-2}(x) = su_{n-3}(x) \text{ for } n \geq 3 \text{ and } u_{-1}(x) = 1 \text{ for } n = 2. \tag{4.7}$$

From (4.6) and (4.7) it follows that the formula (4.8) holds for the recurrence sequence $u_n(x)$ defined by (4.4) and (4.5):

$$u_{n-1}^2(x) - u_n(x)u_{n-2}(x) = (\det A(x))^n \tag{4.8}$$

for every natural number $n \geq 2$. Now, we deduce from (4.8) the Horadam-Mahon formula for Pell polynomials. Indeed, let $a(x) = d(x) = x$ and $b(x) = c(x) = \sqrt{x^2+1}$. Then the matrix $A(x) = P(x)$ has the form

$$P(x) = \begin{pmatrix} x & \sqrt{x^2+1} \\ \sqrt{x^2+1} & x \end{pmatrix}, \tag{4.9}$$

and the recurrence sequence $P_n(x)$ associated with the matrix $P(x)$ satisfies the following conditions:

$$r = TrP(x) = 2x, \;\; s = -\det P(x) = 1, \tag{4.10}$$

and

$$P_n(x) = rP_{n-1}(x) + sP_{n-2}(x) = 2xP_{n-1}(x) + P_{n-2}(x). \tag{4.11}$$

Here, $P_n(x)$ denotes the Pell polynomial. Replacing $u_n(x)$ by $P_n(x)$ in the formula (4.8) we obtain the Horadam-Mahon formula for Pell polynomials.

In the same way we produce more general formulas connected with classes of polynomials $W_n(x; b, c)$ considered in Theorem 2. Namely, we have the following:

Proposition 1: Let $W(x; b, c) = \begin{pmatrix} x & \sqrt{x^2+bx+c} \\ \sqrt{x^2+bx+c} & x \end{pmatrix}$ be a 2×2 functional matrix and let $W_n(x; b, c)$ be the functional recurrence sequence associated with the matrix $W(x; b, c)$ defined by the formulas:

$$\begin{aligned} r = TrW(x; b, c) = 2x, \;\; s &= -\det W(x; b, c) \\ &= -(x^2 - (x^2 + bx + c)) = bx + c \end{aligned}$$

and

$$W_0(x;b,c) = r = 2x,\ W_1(x;b,c) = rW_0(x;b,c) + s = 4x^2 + bx + c$$

and for $n \geq 2$

$$W_n(x;b,c) = rW_{n-1}(x;b,c) + sW_{n-2}(x;b,c) = 2xW_{n-1}(x;b,c) + (bx+c)W_{n-2}(x;b,c).$$

Then for every natural number $n \geq 2$ we have

$$W_{n-1}^2(x;b,c) - W_{n-2}(x;b,c)W_n(x;b,c) = (\det W(x;b,c))^n = (-1)^n(bx+c)^n.$$

Proof: In the first step, by inductive manner as in [1], (pages 116-117), we obtain an analog of formula (4.6) for the powers of the matrix $W(x;b,c)$, using the recurrence sequence $W_n(x;b,c)$. The final step relies on applying Cauchy's theorem on product of determinants. □

In a similar way as in [1], (pages 118-119) we obtain the following:

Proposition 2: Let k be a non-zero constant and let $a = a(x)$ and $b = b(x)$ be given functions of the variable x. Then for every natural number n we have

$$\begin{pmatrix} a(x) & b(x) \\ kb(x) & a(x) \end{pmatrix}^n = \begin{pmatrix} R_n(x) & S_n(x) \\ kS_n(x) & R_n(x) \end{pmatrix},$$

where

$$R_n(x) = \frac{1}{2}\left(\left(a(x) + b(x)\sqrt{k}\right)^n + \left(a(x) - b(x)\sqrt{k}\right)^n\right)$$

and

$$S_n(x) = \frac{1}{2\sqrt{k}}\left(\left(a(x) + b(x)\sqrt{k}\right)^n - \left(a(x) - b(x)\sqrt{k}\right)^n\right). \quad \square$$

Putting $k = 1$ in the last equalities we obtain an explicit connection between the functions $u(x) = a(x) - b(x)$ and $v(x) = a(x) + b(x)$ considered in Theorem 2 with powers of the functional matrices and the corresponding functional recurrences.

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