VIETA CONVOLUTIONS AND DIAGONAL POLYNOMIALS

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1. INTRODUCTION. VIETA DIPTYCH

Two separate, but related, matters are discussed in this communication. One presents a few basic properties of Vieta convolutions, the other offers an outline of the main features of rising and falling diagonal polynomial functions for the Vieta polynomials.

Vieta polynomials are of two kinds [7], the *Vieta-Fibonacci polynomials* $V_n(x)$ and the *Vieta-Lucas polynomials* $v_n(x)$, defined for our purposes by generating functions as, respectively,

$$\sum_{n=1}^{\infty} V_n(x)y^{n-1} = [1 - xy + y^2]^{-1}, \quad V_0(x) = 0, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} v_n(x)y^n = (2 - xy)[1 - xy + y^2]^{-1}. \tag{1.2}$$

Combinatorial, Binet form and recurrence definitions of $V_n(x)$ and $v_n(x)$, along with many detailed properties of these polynomials, are provided in [7]. One might also consult [14] for other facets of $V_n(x)$. Vieta polynomials are so named to honour the French mathematician Vieta (Francois Viète, 1540-1603.)

A Value of Convolutions

Why do we give emphasis to a study of convolutions defined in terms of generating functions?

Looking at (1.1) and (2.1), we see immediately that $V_n(x)$ is a special case of $V_n^{(k)}(x)$ when $k = 0$. Viewed reversely, $V_n^{(k)}(x)$ is a generalization of $V_n(x)$. For the author, the importance of a study of convolutions lies in this dual perspective.

Similar comments apply to $v_n(x)$ (1.2) and $v_n^{(k)}(x)$ (2.8).

2. VIETA CONVOLUTIONS

Vieta-Fibonacci Convolutions

Definition: The k^{th} *Vieta-Fibonacci convolution* $V_n^{(k)}(x)$ of $V_n(x)$ is generated by

$$\sum_{n=1}^{\infty} V_n^{(k)}(x) y^{n-1} = [1 - xy + y^2]^{-(k+1)}, \quad V_0^{(k)}(x) = 0. \tag{2.1}$$

For the explicit representation of the polynomials $V_n^{(k)}(x)$ see Theorem 2 (2.17) and Theorem 1 (2.16) when $k = 1$.

Examples:

$$\begin{aligned} &V_1^{(1)}(x) = 1, V_2^{(1)}(x) = 2x, V_3^{(1)}(x) = 3x^2 - 2, V_4^{(1)}(x) = 4x^3 - 6x, \\ &V_5^{(1)}(x) = 5x^4 - 12x^2 + 3, V_6^{(1)}(x) = 6x^5 - 20x^3 + 12x, \ldots. \end{aligned} \tag{2.2}$$

Evaluation of higher order convolutions ($k \geq 2$) is left to the inclination of the reader. Note that $V_n^{(0)}(x) = V_n(x)$ by (1.1), (2.1).

Basic Properties of $V_n^{(k)}(x)$

Immediately from (2.1)

$$V_n^{(k-1)}(x) = V_n^{(k)}(x) - xV_{n-1}^{(k)}(x) + V_{n-2}^{(k)}(x) \quad (k \geq 1, \ n \geq 2). \tag{2.3}$$

Differentiate (2.1) partially with respect to y after replacing k by $k - 1$. Then

$$(n-1)V_n^{(k-1)}(x) = k\left(xV_{n-1}^{(k)}(x) - 2V_{n-2}^{(k)}(x)\right). \tag{2.4}$$

Eliminate $V_n^{(k-1)}(x)$ from (2.3) and (2.4) to derive

$$(n-1)V_n^{(k)}(x) = (n+k-1)xV_{n-1}^{(k)}(x) - (n+2k-1)V_{n-2}^{(k)}(x). \tag{2.5}$$

Now write

$$\frac{\partial}{\partial x}V_n(x) \equiv V_n'(x), \frac{\partial^2}{\partial x^2}V_n(x) \equiv V_n''(x), \ldots, \frac{\partial^k}{\partial x^k}V_n(x) \equiv V_n^k(x). \tag{2.6}$$

Differentiating (1.1) k-times with respect to y, we arrive at the neat result

$$V_n^k(x) = k!V_{n-k}^{(k)}(x). \tag{2.7}$$

Vieta-Lucas Convolutions

Definition: The k^{th} *Vieta-Lucas convolution* $v_n^{(k)}(x)$ of $v_n(x)$ is generated by

$$\sum_{n=0}^{\infty} v_n^{(k)}(x)y^n = (2-xy)^{k+1}[1-xy+y^2]^{-(k+1)} \tag{2.8}$$

so that $v_n^{(0)}(x) = v_n(x)$ by (1.2), (2.8).

For the explicit representation of the polynomials $v_n^{(k)}(x)$ see Theorem 3 (2.19).

Examples:

$$\begin{aligned} &v_0^{(1)}(x) = 4, v_1^{(1)}(x) = 4x, v_2^{(1)}(x) = 5x^2 - 8, v_3^{(1)}(x) = 6x^3 - 16x, \\ &v_4^{(1)}(x) = 7x^4 - 26x^2 + 12, v_5^{(1)}(x) = 8x^5 - 38x^3 + 36x, \ldots. \end{aligned} \tag{2.9}$$

Because of the nature of the complicated algebra involved (unappetizing mental pabulum!), we restrict our treatment to the simplest case $k = 1$.

Basic Properties of $v_n^{(k)}(x)(k = 1)$

Proceeding similary as in (2.3)-(2.5) for $V_n(x)$, we extract the following essential relationships:

$$v_{n-1}^{(1)}(x) = 4V_n^{(1)}(x) - 4xV_{n-1}^{(1)}(x) + x^2V_{n-2}^{(1)}(x), \tag{2.10}$$

$$nv_n(x) = xV_n^{(1)}(x) - 4V_{n-1}^{(1)}(x) + xV_{n-2}^{(1)}(x), \tag{2.11}$$

$$nxv_n(x) = (x^2-4)V_n^{(1)}(x) + v_{n-1}^{(1)}(x). \tag{2.12}$$

Observe the rather different sorts of equations (2.10)-(2.12) here compared with those in (2.3)-(2.5), as a consequence of the primacy and simplicity of the generating function for $V_n^{(1)}(x)$.

Lastly, if we multiply numerator and denominator of (2.8) when $k = 0$ by $1 - xy + y^2$, then the ensuing algebra reduces to

$$v_{n-1}(x) = 2V_n^{(1)}(x) - 3xV_{n-1}^{(1)}(x) + (2+x^2)V_{n-2}^{(1)}(x) - xV_{n-3}^{(1)}(x). \tag{2.13}$$

Closed Forms

Lemma 1:

$$\binom{N-r}{1}\binom{N-r-1}{r}+2\binom{N-r}{1}\binom{N-r-1}{r-1}=N\binom{N-r}{r}. \tag{2.14}$$

Lemma 2:

$$k\left\{\binom{N+k-1-r}{k}\binom{N-r-1}{r}+2\binom{N+k-1-r}{k}\binom{N-r-1}{r-1}\right\}$$

$$=N\binom{N+k-1-r}{k-1}\binom{N-r}{r}. \tag{2.15}$$

Both lemmas are readily established by routine combinatorial calculation. Clearly, Lemma 1 is a special case of Lemma 2 occurring when $k=1$. Observe that in (2.15), the factor k is absorbed into the product and N emerges as a factor. (See also [8, (2.11a), (4.12a)] where the same two formulas (2.14) and (2.15) appear.)

Theorem 1:

$$V_n^{(1)}(x)=\sum_{r=0}^{[\frac{n-1}{2}]}(-1)^r\binom{n-r}{1}\binom{n-r-1}{r}x^{n-2r-1}. \tag{2.16}$$

Proof (by induction):

The theorem is verifiably valid for $n=1,2,3$ (say). Assume that it is true for $n=N$, that is, assume

$$V_N^{(1)}(x)=\sum_{r=0}^{[\frac{N-1}{2}]}(-1)^r\binom{N-r}{1}\binom{N-r-1}{r}x^{N-2r-1}. \quad (A)$$

Then, with $n \to n+1$, the right-hand side of (2.5) transforms to

$$N\left[xV_N^{(1)}(x) - V_{N-1}^{(1)}(x)\right] + \left[xV_N^{(1)}(x) = 2V_{N-1}^{(1)}(x)\right]$$

$$= N\sum_{r=0}^{[\frac{N}{2}]}(-1)^r\binom{N-r}{1}\binom{N-r}{r}x^{N-2r} + N\sum_{r=0}^{[\frac{N}{2}]}\binom{N-r}{r}x^{N-2r} \qquad \text{by } (A), \text{Lemma 1}$$

$$= N\sum_{r=0}^{[\frac{N}{2}]}(-1)^r\binom{N-r+1}{1}\binom{N-r}{r}x^{N-2r} \qquad (B)$$

$$= NV_{N+1}^{(1)}(x) \qquad (C)$$

which must be the left-hand side of (2.5).

Consequently, (B) and (C) together with (A) reveal that (2.16) is true for all values of n. Accordingly, Theorem 1 is fully established.

Theorem 2:

$$V_n^{(k)}(x) = \sum_{r=0}^{[\frac{n-1}{2}]}(-1)^r\binom{n+k-r-1}{k}\binom{n-r-1}{r}x^{n-2r-1}. \tag{2.17}$$

Proof (by Induction): Follow the procedures in the proof of Theorem 1 while utilizing Lemma 2. (Pascal's Formula is needed in both Theorems 1 and 2.)

Examples:

$$V_1^{(k)} = 1, \quad V_2^{(k)}(x) = \binom{k+1}{1}x, \quad V_3^{(k)}(x) = \binom{k+2}{2}x^2 - \binom{k+1}{1},$$

$$V_4^{(k)}x = \binom{k+3}{3}x^3 - 2\binom{k+2}{2}x, \ldots, \tag{2.18}$$

as may be checked by (2.1).

By virtue of the generating functions (2.1) and (2.8) for $V_n^{(k)}(x)$ and $v_n^{(k)}(x)$ respectively, and in view of Theorem 2, it is clear that $v_n^{(k)}(x)$ may be expressed combinatorially in summation form involving the Vieta convolutions.

Theorem 3:

$$v_n^{(k)}(x) = \sum_{r=0}^{k+1}(-1)^r \binom{k+1}{r} 2^{k+1-r} x^r V_{n-r+1}^{(k)}(x), \tag{2.19}$$

where $V_{n-r+1}^{(k)}(x)$ are given in (2.17).

Proof: Expand $(2-xy)^{k+1}$ in conjunction with (2.1) and (2.8). Theorem 3, as enunciated, then follows.

Examples:

$$v_0^{(k)}(x) = 2^{k+1}, \quad v_1^{(k)}(x) = 2^k \binom{k+1}{1} x,$$

$$v_2^{(k)}(x) = 2^{k-1}\left[4\binom{k+2}{2} - 2\binom{k+1}{1}^2 + \binom{k+1}{2}\right] x^2 - 2^{k+1}\binom{k+1}{1}, \ldots \tag{2.20}$$

Putting $k = 1$ in (2.20) reduces these expressions to those in (2.9). Theorem 1 corresponds to Theorem 3 when $k = 1$.

A Question Answered.

In [7], some elegant results connecting Vieta, Jacobsthal, and Morgan-Voyce polynomials with special arguments $\frac{1}{x}$, $-x^2$, $-\frac{1}{x^2}$ were revealed. Note that in the definitions of Jacobsthal polynomials $J_n(x)$ and Jacobsthal-Lucas polynomials $j_n(x)$ given in [6] and [8], *the factor* $2x$ *is here replaced by* x as in [7].

At the Luxembourg International Fibonacci Conference (July, 2000) the question was asked:

Can these special results be cárried over to convolution theory?

Sadly, the answer is: generally, NO!

Happily, however, there is one positive instance, namely,

Theorem 4:

$$V_n^{(k)}(x) = x^{n-1} J_n^{(k)}\left(-\frac{1}{x^2}\right). \tag{2.21}$$

Proof:

(a) By Theorem 2 and [7, Theorem 1], both expressions are equal to the combinatorial summation

$$\sum_{r=0}^{[\frac{n-1}{2}]} (-1)^r \binom{n+k-r-1}{k} \binom{n-r-1}{r} x^{n-2r-1},$$

with the same initial values 0 and 1 for $n = 0, 1$.

(b) Working from the recurrence relation [8, (4.13)] for $x^{n-1} J_n^{(k)} \left(-\frac{1}{x^2}\right)$ we quickly have, on multiplying throughout by x^{n-1},

$$x^{n+1} J_{n+2}^{(k)} \left(-\frac{1}{x^2}\right) - x \cdot x^n J_{n+1}^{(k)} \left(-\frac{1}{x^2}\right) + x^{n-1} J_n^{(k)} \left(-\frac{1}{x^2}\right) = x^{n+1} J_{n+2}^{(k)} \left(-\frac{1}{x^2}\right)$$

which is identical with (2.3) for $x^{n-1} J_n^{(k)} \left(-\frac{1}{x^2}\right) = V_n^{(k)}(x)$, both of which have initial values 0 and 1 for $n = 0, 1$.

Note:

(i) An analysis of the expansion of the generating function $\left[1 - y - \frac{y^2}{x^2}\right]^{-2}$ for $J_{n+1}^{(1)} \left(-\frac{1}{x^2}\right)$

leads us to a verification of Theorem 3 for $V_n^{(1)}(x)$, for small values of n.

(ii) No such joys as in Theorem 4 await us when we turn to $v_n^{(k)}(x)$ and $j_n^{(k)} \left(-\frac{1}{x^2}\right)$, as is evident from the more complicated forms of their generating functions.

Coming to Morgan-Voyce convolutions, we find there is no connection with Vieta and Jacobsthal convolutions for the above special arguments, since the essential provisos in the Proofs in Theorem 4 do not pertain. [Parenthetically, we remark that even the beautiful Cinderella had less attractive sisters!]

Cauchy Product

Convolution polynomials $V_n^{(i)}(x)(i = 1, \ldots, k)$ may also be defined by means of summations of **Cauchy products**, thus:

Definition:

$$
\begin{aligned}
V_n^{(1)}(x) &= \sum_{r=1}^{n} V_r(x)V_{n+1-r}(x), \\
V_n^{(2)}(x) &= \sum_{r=1}^{n} V_r^{(1)}(x)V_{n+1-r}(x), \\
\dots \quad & \quad \dots \\
V_n^{(k)}(x) &= \sum_{r-1}^{n} V_r^{k-1}(x)V_{n+1-r}(x).
\end{aligned}
\tag{2.22}
$$

Examples:

$$
\begin{aligned}
V_5^{(1)}(x) &= 2V_1(x)V_5(x) + 2V_2V_4(x) + (V_3(x))^2 = 5x^4 - 12x^2 + 3 \text{ as in (2.2), Theorem 1.} \\
V_4^{(2)}(x) &= V_1^{(1)}(x)V_4(x) + V_2^{(1)}(x)V_3(x) + V_3^{(1)}(x)V_2(x) + V_4^{(1)}(x)V_1(x) \\
&= 10x^3 - 12x \quad \text{as from (2.18), } k = 2.
\end{aligned}
$$

Cauchy products may likewise define the Vieta-Lucas convolution polynomials $v_n^{(i)}(x)(i = 1, \dots, k)$.

Definition:

$$
\begin{aligned}
v_n^{(1)}(x) &= \sum_{r=0}^{n} v_r(x)v_{n-r}(x), \\
v_n^{(2)}(x) &= \sum_{r=0}^{n} v_r^{(1)}(x)v_{n-r}(x), \\
\dots \quad & \quad \dots \\
v_n^{(k)}(x) &= \sum_{r=0}^{n} v_r^{k-1}(x)v_{n-r}(x).
\end{aligned}
\tag{2.23}
$$

Examples:

$$
v_5^{(1)}(x) = 2v_0(x)v_4(x) + 2v_1(x)v_3(x) + (v_2(x))^2 = 7x^4 - 26x^2 + 12
$$

as in (2.9)

$$\begin{aligned} v_3^{(2)}(x) &= v_0^{(1)}(x)v_3(x) + v_1^{(1)}(x)v_2(x) + v_2^{(1)}(x)v_1(x) + v_3^{(1)}(x)v_0(x) \\ &= 25x^3 - 60, \quad \text{as from (2.20)}, \; k = 2. \end{aligned}$$

Thus, there exist three ways of calculating, say, $v_2^{(2)}(x) = 18x^2 - 24$, namely: (i) directly from (2.8), $k = 2, n = 2$ (ii) by substituting $k = 2, n = 2$ in (2.20) [equivalent really to (i)], and (iii) by using the Cauchy product (2.23).

Remarks:

(a) Generally, we may extend (2.22) to

$$V_n^{(k)}(x) = \sum_{r=1}^{n} V_r^{(m)}(x) V_{n+1-r}^{(k-1-m)}(x) \; (m = 0, 1, \ldots, k-1). \tag{2.24}$$

Likewise for $v_n^{(k)}(x)$.

(b) Cauchy products as in (2.22-2.24) are applicable analogously to Jacobsthal-type polynomials [8], Morgan-Voyce polynomials [9], Fermat-type polynomials [10], and to Pell and Pell-Lucas polynomials (for which see A.F. Horadam and Bro. J.M. Mahon: "Convolutions for Pell Polynomials." *Fibonacci Numbers and Their Applications* (Eds. A.N. Philippou, G.E. Bergum, and A.F. Horadam), Kluwer Academic Publishers, Dordrecht, The Netherlands (1986): 55-80).

Variation on a Theme

Suppose we replace $+y^2$ by $-y^2$ in (2.1) and (2.8). Designate the ensuing modified polynomials by $*V_n^{(k)}(x)$ and $*v_n^{(k)}(x)$ respectively. Of course, it then transpires that

$$*V_n^{(k)}(x) = F_n^{(k)}(x), \; *v_n^{(k)}(x) = L_n^{(k)}(x), \tag{2.25}$$

where $F_n^{(k)}(x)$ and $L_n^{(k)}(x)$ are the generalized Fibonacci and Lucas k^{th} convolution polynomials, respectively. In fact, for example, $*V_6^{(1)}(x) = 6x^5 + 20x^3 + 12x$.

Mindful that $*V_n^{(0)}(1) = F_n$, the n^{th} Fibonacci number, we may build up the Fibonacci convolution sequences as, e.g.,

$$\begin{aligned} \{*V_n^{(1)}(1)\} &= \{F_n^{(1)}\} = \{1, 2, 5, 10, 20, 38, 71, 130, \ldots\}, \\ \{*V_n^{(2)}(1)\} &= \{F_n^{(2)}\} = \{1, 3, 9, 22, 51, 111, 233, \ldots\}, \\ \{*V_n^{(3)}(1)\} &= \{F_n^{(3)}\} = \{1, 4, 14, 40, 105, 246, 594, \ldots\}, \end{aligned} \tag{2.26}$$

which may, for visual convenience, be expressed in tabular form. Calculations in (2.26) have involved (2.5), (2.17), and (2.18). Verfications may be obtained by recourse to V.E. Hoggatt, Jr. and G.E. Bergum, "Generalized Convolution Arrays", *The Fibonacci Quarterly* 13.3 (1975): 193-196. Sequences occurring in (2.26) appear in the table on page 118 of V.E. Hoggatt, Jr. and Marjorie Bicknell-Johnson, "Fibonacci Convolution Sequences", *The Fibonacci Quarterly* 15.2 (1977): 117-122.

3. VIETA DIAGONAL POLYNOMIALS

Preamble

While sorting out ideas on rising and falling diagonal functions for $V_n(x)$ and $v_n(x)$, the author became aware of the generalized survey in [15] covering similar work already done for Fibonacci, Lucas, Chebyshev [1], [3], [12], Fermat [3], and Jacobsthal [6] polynomials.

To these polynomials we specifically add the earlier study of Pell polynomials [13] and Gegenbauer polynomials [11] (rising diagonals) and [5] (descending diagonals). Work on Morgan-Voyce rising and descending diagonal polynomials is under investigation.

Each polynomial has an individual essence distinguishing it from others. Our justification for treating Vieta diagonal polynomials as separate entities and not just as particular instances of a general situation is that it preserves the distinguishing features of these polynomials and so it enhances our knowledge of Vieta polynomials *per se*.

The slanting criss-cross pattern of rising and falling parallel diagonal "lines" is visually apparent for the polynomials displayed in [2], [3], [4], and [11]. Incidentally, both kinds of Chebyshev polynomials are special cases of Gegenbauer polynomials [5, p. 294], [11, p. 394].

Rising Vieta Diagonal Polynomials

Represent these polynomials for $V_n(x)$ and $v_n(x)$ by $R_n(x)$ and $r_n(x)$ respectively. Then the following fundamental conclusions are relatively easy to establish.

Generating Functions

$$\sum_{n=1}^{\infty} R_n(x)y^{n-1} = [1-y(x-y^2)]^{-1}, \quad R_0(x)=0. \tag{3.1}$$

$$\sum_{n=3}^{\infty} r_n(x)y^{n-1} = (1-y^3)[1-y(x-y^2)]^{-1}, \quad r_0(x)=2. \tag{3.2}$$

Recurrence Relations

$$R_n(x) = xR_{n-1}(x) - R_{n-3}(x). \tag{3.3}$$

$$r_n(x) = xr_{n-1}(x) - r_{n-3}(x). \tag{3.4}$$

$$r_n(x) = R_n(x) - R_{n-3}(x). \tag{3.5}$$

Computation of the $R_n(x)$ and $r_n(x)$ in (3.1) and (3.2) is left to the dedication of the reader.

Descending Vieta Diagonal Polynomials

Designate these polynomials for $V_n(x)$ and $v_n(x)$ by $D_n(x)$ and $d_n(x)$ respectively. Analogues of the generating functions and recurrence relations for $R_n(x)$ and $r_n(x)$ are straightforward to discover.

Generating Functions

$$D_n(x) = (x-1)^{n-1}, \quad D_0(x) = 0, \tag{3.6}$$

$$d_n(x) = (x-2)(x-1)^{n-1}, \quad d_0(x) = 2, \tag{3.7}$$

whence

$$\frac{d_n(x)}{D_n(x)} = x - 2. \tag{3.8}$$

Recurrence Relations

$$\frac{d_n(x)}{d_{n-1}(x)} = \frac{D_n(x)}{D_{n-1}(x)} = x - 1. \tag{3.9}$$

Partial Differentiation

Suppose now that we use the generating function symbolism

$$G \equiv G(x,y) = [1-(x-1)y]^{-1} = \sum_{n=1}^{\infty} D_n(x)y^{n-1}. \tag{3.10}$$

An immediate outcome is that

$$(x-1)\frac{\partial G}{\partial x} = y\frac{\partial G}{\partial y}. \tag{3.11}$$

Setting

$$H \equiv H(x,y) = (x-2)[1-(x-1)y]^{-1} = \sum_{n=1}^{\infty} d_n(x)y^n. \tag{3.12}$$

we come to

$$(x-1)(x-2)\frac{\partial H}{\partial x} = (1-y)\frac{\partial H}{\partial y}. \tag{3.13}$$

Partial differentiation along the procedures of (3.10) - (3.13) for $R_n(x)$ and $r_n(x)$ is a suggested exercise.

4. CONCLUSION

In passing, we mention that the 1969 formula occurring in [7, reference [1], p. 14],

$$v_n(p,q) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} p^{n-2k} q^k,$$

and surely of an earlier origin, is equivalent to the 1999 formula [15, (2.22)] when $x = p,\ y = -q$. Attention to the valuable material in [15] is strongly recommended.

Attention might also be directed to the related study of convolutions for generalized Fibonacci and Lucas Polynomials in [10].

The purpose of this paper has been to give a skeletal framework to the theory which, hopefully, could be fleshed out to a more robust body of knowledge.

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