A THREE-VARIABLE IDENTITY INVOLVING CUBES OF FIBONACCI NUMBERS

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1. INTRODUCTION

The identities

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \tag{1.1}$$

and

$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n} \tag{1.2}$$

are special cases of identity (5) of Torretto and Fuchs [7]. Interestingly, (1.2) is the only identity involving cubes of Fibonacci numbers that appears in Dickson's *History of the Theory of Numbers* [1, p. 395], and Dickson attributes it to Lucas.

In [6], the following generalizations of (1.1) and (1.2), together with their Lucas counterparts, were given.

$$F_{n+k+1}^2 + F_{n-k}^2 = F_{2k+1}F_{2n+1}; \tag{1.3}$$

$$F_{3k+1}F_{n+k+1}^3 + F_{3k+2}F_{n+k}^3 - F_{n-2k-1}^3 = F_{3k+1}F_{3k+2}F_{3n}. \tag{1.4}$$

In fact, as was proved by Howard [5], (1.3) is equivalent to

$$F_n^2 + (-1)^{n+k+1}F_k^2 = F_{n-k}F_{n+k}, \tag{1.5}$$

occurring as I_{19} on page 59 in [4]. In (1.5), replacing n by $n+k$, and k by n yields

$$F_{n+k}^2 + (-1)^{k+1}F_n^2 = F_kF_{2n+k}, \tag{1.6}$$

equivalent to (1.5), and which we require in the sequel.

Recently, we were made aware of the identity

$$F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n} \tag{1.7}$$

due to Ginsburg [3], and this prompted us to search for a more general identity that yields (1.2), (1.4), and (1.7) as special cases. This identity is stated in the next section, and our proof of it relies on a powerful method given recently by Dresel [2]. For instance, in the terminology of Dresel, (1.1) is *homogeneous* of degree 2 in the variable n. As such, to prove it we need only verify its validity for 3 distinct values of n.

Quite often, after discovering a new Fibonacci identity, we expend energy trying to discover its Lucas counterpart. Dresel's *duality theorem* provides us with a way of achieving this quickly. Indeed, the duality theorem produces a *dual* identity for *any* homogeneous Fibonacci-Lucas (FL) identity.

The Duality Theorem (Dresel): Given a homogeneous FL-identity in the variable n, we can arrive at a new dual identity with respect to the variable n by making the following changes throughout:

(i) when j is odd, F_{jn+k} is replaced by $L_{jn+k}/\sqrt{5}$,

(ii) when j is odd, L_{jn+k} is replaced by $\sqrt{5}F_{jn+k}$,

(iii) when j is odd, $(-1)^{jn}$ is replaced by $-(-1)^{jn}$.

The justification for each step in the theorem is easily seen if we refer to the Binet forms. For example, the dual of (1.1) is $L_{n+1}^2 + L_n^2 = 5F_{2n+1}$. We give further illustrations after the proof of our main result, when we employ the duality theorem to produce seven additional identities.

2. THE MAIN RESULT

We make use of the following identities.

$$F_{-n} = (-1)^{n+1}F_n, \tag{2.1}$$

$$F_{n+k} + F_{n-k} = L_n F_k, \quad k \text{ odd}, \tag{2.2}$$

$$F_{n+k} - F_{n-k} = L_n F_k, \quad k \text{ even}, \tag{2.3}$$

$$F_{2n} = F_n L_n, \tag{2.4}$$

$$(-1)^{k+1}F_k F_{n+k}^3 - F_k F_{n-k}^3 + F_{2k}F_n^3 = (-1)^{k+1}F_k^2 F_{2k} F_{3n}. \tag{2.5}$$

Identities (2.1) and (2.4) are well known, while identities (2.2) and (2.3) occur as I_{22} and I_{24}, respectively, on page 59 in [4]. Identity (2.5), which appears as (5.2) in [2], can be expressed more simply if we factor out F_k. However, in its present form, its relationship with our main result is more transparent. Our main result follows.

Theorem: Let k, m, and n be any integers. Then

$$F_m F_{n+k}^3 + (-1)^{k+m+1}F_k F_{n+m}^3 + (-1)^{k+m}F_{k-m}F_n^3 = F_{k-m}F_k F_m F_{3n+k+m}. \tag{2.6}$$

Proof: Since (2.6) is homogeneous of degree 3 in the variable n, we need only verify its validity for four distinct values of n. If $k = m$, or if one of k or m is zero, then (2.6) follows immediately. Furthermore, if $k + m = 0$, then (2.6) follows from (2.5). So we may assume that $km(k-m)(k+m) \neq 0$. But then $0, -k, -m$, and $-k-m$ are distinct, and so we need only verify (2.6) for these four values of n. We perform the verifications for $n = -k$ and $n = -k - m$, and leave the remaining verifications to the reader.

Using (2.1), we find that $F^3_{-k+m} = (-1)^{k-m+1}F^3_{k-m}$, and $F^3_{-k} = (-1)^{k+1}F^3_k$. Then, for $n = -k$,

$$\begin{aligned}
LHS &= (-1)^{k+m+1}F_kF^3_{-k+m} + (-1)^{k+m}F_{k-m}F^3_{-k} \\
&= F_kF^3_{k-m} + (-1)^{m+1}F_{k-m}F^3_k \\
&= F_{k-m}F_k\left[F^2_{k-m} + (-1)^{m+1}F^2_k\right] \\
&= F_{k-m}F_k\left[F^2_{k-m} + (-1)^{-m+1}F^2_k\right] \\
&= F_{k-m}F_kF_{-m}F_{2k-m} \qquad \text{(using (1.6))} \\
&= F_{k-m}F_kF_{-m}F_{-(-2k+m)} \\
&= F_{k-m}F_k(-1)^{m+1}F_m(-1)^{-2k+m+1}F_{-2k+m} \qquad \text{(using (2.1))} \\
&= F_{k-m}F_kF_mF_{-2k+m} \\
&= RHS.
\end{aligned}$$

For $n = -k - m$ we have

$$\begin{aligned}
LHS &= F_mF^3_{-m} + (-1)^{k+m+1}F_kF^3_{-k} + (-1)^{k+m}F_{k-m}F^3_{-k-m} \\
&= (-1)^{m+1}F^4_m + (-1)^mF^4_k - F_{k-m}F^3_{k+m} \qquad \text{(using (2.1))} \\
&= (-1)^m\left[F^4_k - F^4_m\right] - F_{k-m}F^3_{k+m} \\
&= (-1)^m\left[F^2_k + (-1)^{k+m+1}F^2_m\right]\left[F^2_k + (-1)^{k+m}F^2_m\right] - F_{k-m}F^3_{k+m} \\
&= (-1)^m\left[F^2_{m+(k-m)} + (-1)^{k-m+1}F^2_m\right]\left[F^2_k + (-1)^{k+m}F^2_m\right] - F_{k-m}F^3_{k+m} \\
&= (-1)^mF_{k-m}F_{k+m}\left[F^2_k + (-1)^{k+m}F^2_m\right] - F_{k-m}F^3_{k+m} \qquad \text{(using (1.6))} \\
&= F_{k-m}F_{k+m}\left[(-1)^mF^2_k - \left[F^2_{m+k} + (-1)^{k+1}F^2_m\right]\right] \\
&= F_{k-m}F_{k+m}\left[(-1)^mF^2_k - F_kF_{2m+k}\right] \qquad \text{(using (1.6))} \\
&= -F_{k-m}F_{k+m}F_k\left[F_{(m+k)+m} + (-1)^{m+1}F_{(m+k)-m}\right] \\
&= -F_{k-m}F_{k+m}F_kL_{k+m}F_m \qquad \text{(using (2.2) and (2.3))} \\
&= -F_{k-m}F_kF_mF_{2k+2m} \qquad \text{(using (2.4))} \\
&= RHS, \text{ using (2.1)}.
\end{aligned}$$

This completes the proof of the Theorem. □

Now, since (2.6) is homogeneous of degree 3 in the variable n, its dual identity, with respect to n is

$$F_m L_{n+k}^3 + (-1)^{k+m+1} F_k L_{n+m}^3 + (-1)^{k+m} F_{k-m} L_n^3 = 5F_{k-m} F_k F_m L_{3n+k+m}. \tag{2.7}$$

Before proceeding we note that, since $(-1)^k = (\alpha\beta)^k, (-1)^k F_k$ has degree 3 with respect to the variable k. Hence (2.6) and (2.7) are each homogeneous of degree 3 in k, and their duals with respect to k are, respectively,

$$F_m L_{n+k}^3 + 5(-1)^{k+m} L_k F_{n+m}^3 + 5(-1)^{k+m+1} L_{k-m} F_n^3 = L_{k-m} L_k F_m L_{3n+k+m}, \tag{2.8}$$

and

$$25F_m F_{n+k}^3 + (-1)^{k+m} L_k L_{n+m}^3 + (-1)^{k+m+1} L_{k-m} L_n^3 = 5L_{k-m} L_k F_m F_{3n+k+m}. \tag{2.9}$$

Finally, since $F_m = (-1)^{2m} F_m, F_{k-m} = (-1)^{m-k+1} F_{m-k}$, and $L_{k-m} = (-1)^{m-k} L_{m-k}$, we see that (2.6)-(2.9) are each homogeneous of degree 5 in m. Accordingly, we find that their duals in the variable m are, respectively,

$$5L_m F_{n+k}^3 + (-1)^{k+m} F_k L_{n+m}^3 + 5(-1)^{k+m+1} L_{k-m} F_n^3 = L_{k-m} F_k L_m L_{3n+k+m}, \tag{2.10}$$

$$L_m L_{n+k}^3 + 25(-1)^{k+m} F_k F_{n+m}^3 + (-1)^{k+m+1} L_{k-m} L_n^3 = 5L_{k-m} F_k L_m F_{3n+k+m}, \tag{2.11}$$

$$L_m L_{n+k}^3 + (-1)^{k+m+1} L_k L_{n+m}^3 + 25(-1)^{k+m} F_{k-m} F_n^3 = 5F_{k-m} L_k L_m F_{3n+k+m}, \tag{2.12}$$

$$25L_m F_{n+k}^3 + 25(-1)^{k+m+1} L_k F_{n+m}^3 + 5(-1)^{k+m} F_{k-m} L_n^3 = 5F_{k-m} L_k L_m L_{3n+k+m}. \tag{2.13}$$

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