FIBONACCI NUMBERS AND PARTITIONS

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(Submitted April 2001-Final Revision September 2001)

1. INTRODUCTION

In a series of two papers [6] and [7] Slater gave a list of 130 identities of the Rogers-Ramanujan type. In [2] Andrews has introduced a two variable function in order to look for combinatorial interpretations for those identities. In [5] one of us, Santos, gave conjectures for explicit formulas for families of polynomial that can be obtained using Andrews method for 74 identities of Slater's list.

In this paper we are going to prove the conjectures given by Santos in [5] for identities 94 and 99.

We show, also that the family of polynomials $P_n(q)$ related to identity 94 given by

$$\begin{aligned} &P_0(q) = 1, \ P_1(q) = 1 + q + q^2 \\ &P_n(q) = (1 + q + q^{2n})P_{n-1}(q) - qP_{n-2}(q) \end{aligned} \tag{1.1}$$

is the generating function for partitions into at most n parts in which every even smaller than the largest part appears at least once and that the family $T_n(q)$ related to identity 99 given by

$$\begin{aligned} &T_0(q) = 1, \ T_1(q) = 1 + q^2 \\ &T_n(q) = (1 + q + q^{2n})T_{n-1}(q) - qT_{n-2}(q) \end{aligned} \tag{1.2}$$

is the generating function for partitions into at most n parts in which the largest part is even and every even smaller than the largest appears at least once.

In what follows we denote the Fibonacci numbers by F_n where $F_0 = 0; F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 2$, and use the standard notation

$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1})$$

and

$$(A;q)_\infty = \prod_{n=0}^{\infty}(1-Aq^n), \quad |q|<1.$$

We need also the following identities for the Gaussian polynomials

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix} \tag{1.3}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \tag{1.4}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m}\begin{bmatrix} n-1 \\ m \end{bmatrix} \tag{1.5}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q;q)_m}{q;q)_m(q;q)_{n-m}}, \text{ for } 0 \leq m \leq n, \tag{1.6}$$

$$0 \quad \text{otherwise}$$

2. THE FIRST FAMILY OF POLYNOMIALS

We consider now the two variable function associated to identity 94 of Slater [7] which is:

$$f_{94}(q,t) = \sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(t;q^2)_{n+1}(tq;q^2)_{n+1}}. \tag{2.1}$$

From this we have that

$$(1-t)(1-tq)f_{94}(q,t) = 1 + tq^2 f_{94}(q;tq^2)$$

and in order to obtain a recurrence relation from this functional equation we make the following substitution

$$f_{94}(q,t) = \sum_{n=0}^{\infty} P_n t^n.$$

Now we have:

$$(1-t)(1-tq)\sum_{n=0}^{\infty} P_n t^n = 1 + tq^2 \sum_{n=0}^{\infty} P_n (tq^2)^n$$

which implies

$$\sum_{n=0}^{\infty} P_n t^n - \sum_{n=1}^{\infty} P_{n-1} t^n - \sum_{n=1}^{\infty} qP_{n-1} t^n + \sum_{n=2}^{\infty} qP_{n-2} t^n = 1 + \sum_{n=1}^{\infty} q^{2n} P_{n-1} t^n.$$

From this last equation it is easy to see that

$$\begin{aligned} &P_0(q) = 1;\ P_1(q) = 1 + q + q^2 \\ &P_n(q) = (1 + q + q^{2n})P_{n-1}(q) - qP_{n-2}(q). \end{aligned} \tag{2.2}$$

Santos gave in [5] a conjecture $C_n(q)$, for an explicity formula for this family of polynomials:

$$C_n(q) = \sum_{j=-\infty}^{\infty} q^{q^{15j^2+4j}} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum q^{15j^2+14j+3} \begin{bmatrix} 2n+1 \\ n-5j-2 \end{bmatrix}. \tag{2.3}$$

In our next theorem we prove that this conjecture is true.

Theorem 2.1: The family $P_n(q)$ given in (2.2) is equal to $C_n(q)$ given in (2.3).

Proof: Considering that $C_0(q) = 1$ and $C_1(q) = 1 + q + q^2$ we have to show that

$C_n(q) = (1 + q + q^{2n})C_{n-1}(q) - qC_{n-2}(q)$ that is:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n+1 \\ n-5j-2 \end{bmatrix}$$

$$= (1 + q + q^{2n}) \left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix} \right)$$

$$- q \left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix} \right) \tag{2.4}$$

If we apply (1.4) in each expression on the left side of (2.4) we get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j} \begin{bmatrix} 2n \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3} \begin{bmatrix} 2n \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n \\ n-5j-3 \end{bmatrix}$$

Applying now (1.5) to each sum in the expression above and replacing it in (2.4) we get after some cancellations

$$\sum_{j=-\infty}^{\infty} q^{15j^2+j+n} \begin{bmatrix} 2n-1 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+4} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n-1 \\ n-5j-4 \end{bmatrix}$$

$$= \sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix}. \qquad (2.5)$$

Considering the right side of the last expression and applying (1.4) on the first two sums we get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+1+n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix}$$

Applying now (1.5) on the first and third sums on this last expression and making some cancellations we have that the right side of (2.5) is equal to:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-j+n} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+1+n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+9j+j+1+n} \begin{bmatrix} 2n-3 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$

If we take now the left side of (2.5) and apply (1.4) to all sums we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-j+n} \begin{bmatrix} 2n-2 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+n+6} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+24j+2n+9} \begin{bmatrix} 2n-2 \\ n-5j-5 \end{bmatrix}. \tag{2.6}$$

Applying now (1.5) on the first and fifth sums of this last expression and making cancellations with the sums from the right side given in (2.6) we are left with:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-6j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+24j+2n+9} \begin{bmatrix} 2n-2 \\ n-5j-5 \end{bmatrix}.$$

Observing that the third sum cancels the fifth and replacing j by $j+1$ in the last sum we get after using (1.4)

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n+1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2-j+3n-2} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$

which is identically zero by (1.5) completing the proof. □

Next we make a few observations regarding the combinatorics of $P_N(q)$ given in (2.2). Knowing that $P_n(q)$ is the coefficient of t^N in (2.1) that is:

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(1-t)(tq^2;q^2)_n (tq;q^2)_{n+1}}$$

and considering that $n^2+n = 2+4+\cdots+2n$ we can see that the coefficient of t^N in

$$\frac{t^n q^{n^2+n}}{(tq^2;q^2)_n (tq;q^2)_{n+1}}$$

is the generating function for partitions into exactly N parts in which every even smaller than the largest part appears at least once. Because of the factor $(1-t)$ in the denominator we have proved the following theorem:

Theorem 2.2: $P_n(q)$ is the generating function for partitions into at most N parts in which every even smaller than the largest part appears at least once.

To see, now, the connection between the family of polynomials $P_N(q)$ and the Fibonacci numbers we observe first that if we replace q by 1 in (2.2) we have

$$P_0(1) = 1; P_1(1) = 3$$
$$P_n(1) = 3P_{n-1}(1) - P_{n-2}(1)$$

and that for the Fibonacci sequence F_n we have also that $F_2 = 1;\ F_4 = 3$ and

$$F_{2n+2} = 3F_{2n} - F_{2n-2}$$

which allow us to conclude that

$$C_n(1) = P_n(1) = F_{2n+2}$$

and from these considerations we have proved the following:

Theorem 2.3: The total number of partitions into at most N parts in which every even smaller than the largest part appears at least once is equal to F_{2N+2}.

The family given in (2.2) has also an interesting property at $q = -1$. At this point we have

$$P_0(-1) = 1; \; P_1(-1) = 1$$
$$P_n(-1) = P_{n-1}(-1) + P_{n-2}(-1)$$

which tell us that for $q = -1$ we have all the Fibonacci numbers, i.e. $P_n(-1) = F_{n+1}$. In order to be able to see what happens combinatorially at -1 we have to observe that when we change q by $-q$ in (2.1) the only term that changes is $(tq;q^2)_{n+1}$ and that now the coefficient of t^N is going to be just the number of partitions as described in Theorem 2.3 having an even number of odd parts minus the number of partions of that type with an odd number of odd parts. We state this in our next theorem.

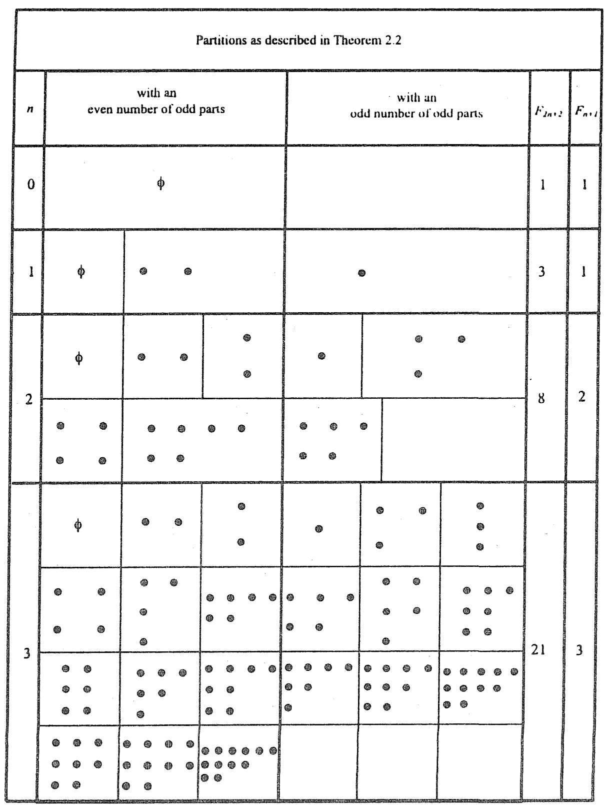

Table 2.1

Theorem 2.4: The total number of partitions into at most N parts in which every even smaller than the largest part appears at least once and having an even number of odd parts minus the number of those with an odd number of odd parts is equal to F_{N+1}.

In the table (2.1) we present, for a few values of n, all the results proved so far. The first column has n, the second the partitions described in theorem 2.4 with an even number of odd parts and the third column those with an odd number of odd parts. The fourth column has F_{2n+2} which is the total number of partitions in columns 2 and 3 and the fifth column has the difference between the number of partitions on the second and third column which is F_{n+1}.

3. THE SECOND FAMILY OF POLYNOMIALS

Now we consider the two variable function given in Santos [5] associated to identity 99 of Slater [7] which is:

$$f_{99}(q,t) = \sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(t;q^2)_{n+1}(tq;q^2)_n} \tag{3.1}$$

From this we can get

$$(1-t)(1-tq)f_{99}(q,t) = 1 - tq + tq^2 f_{99}(q,tq^2)$$

from which we obtain in a way similar to the one used to get (2.2) the following family of polynomials

$$\begin{aligned} &T_0(q) = 1;\ T_1(q) = 1 + q^2 \\ &T_n(q) = (1 + q + q^{2n})T_{n-1}(q) - qT_{n-2}(q) \end{aligned} \tag{3.2}$$

As for the family (2.2) Santos gave in [5] a conjecture for an explicity formula for (3.2) which is

$$B_n(q) = \sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix} \tag{3.3}$$

The proof for this conjecture is given in the next theorem.

Theorem 3.1: The family $T_n(q)$ given in (3.2) is equal to $B_n(q)$ given in (3.3).

Proof: Considering that $B_0(q) = 1$ and $B_1(q) = 1 + q^2$ we have to show that $B_n(q) = (1 + q + q^{2n})B_{n-1}(q) - qB_{n-2}(q)$ which is:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix}$$

$$= (1+q+q^{2n}) \left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} \right)$$

$$- q \left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n-3 \\ n-5j-3 \end{bmatrix} \right) \tag{3.4}$$

We apply (1.4) on each sum on the left to get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1} \begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+2} \begin{bmatrix} 2n \\ n-5j-2 \end{bmatrix}$$

Applying now, (1.5) in all sums we obtain:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+3j} \begin{bmatrix} 2n-1 \\ n-5j \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+2} \begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}$$

Replacing this in (3.4) and making cancellations we are left with:

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n}\begin{bmatrix}2n-1\\ n-5j\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix}2n-1\\ n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+3j+n}\begin{bmatrix}2n-1\\ n-5j-1\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+3}\begin{bmatrix}2n-1\\ n-5j-3\end{bmatrix}\\
&=\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-1\\ n-5j-1\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix}2n-1\\ n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-3\\ n-5j-2\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix}2n-3\\ n-5j-3\end{bmatrix}
\end{aligned} \tag{3.5}$$

Applying (1.4) on the first two sums on the right side of this last expression we get for that side:

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-2\\ n-5j-1\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix}2n-2\\ n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+8j}\begin{bmatrix}2n-2\\ n-5j-2\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+1}\begin{bmatrix}2n-2\\ n-5j-3\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix}2n-3\\ n-5j-2\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix}2n-3\\ n-5j-3\end{bmatrix}
\end{aligned}$$

Using (1.5) on the first and third sums we get after cancellations

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n}\begin{bmatrix}2n-3\\ n-5j-1\end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix}2n-2\\ n-5j-2\end{bmatrix}\\
&-\sum_{j=-\infty}^{\infty} q^{15j^2+3j+n}\begin{bmatrix}2n-3\\ n-5j-2\end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+2+n}\begin{bmatrix}2n-2\\ n-5j-3\end{bmatrix}
\end{aligned}$$

Applying (1.4) in all sums on the left side of (3.5) and making cancellations with the corresponding sums on the right we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n} \begin{bmatrix} 2n-2 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+12j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+18j+2n+4} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$

$$= \sum_{j=-\infty}^{\infty} q^{15j^2-3j+n} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix}$$

Using (1.5) on the first and fourth sums on the LHS we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-8j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+12j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2-2j+2n-1} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+18j+5} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix} = 0.$$

Replacing j by $j-1$ in the last sum and using (1.3) that sum cancels with the third.

If we replace j by $-j$ in the fourth sum using (1.3) and subtract from the second by (1.4) we get finally:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-8j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2-3j+3n-2} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} = 0.$$

To see that this expression is, in fact, identically zero we apply (1.4) on the first two sums replacing j by $-j$ and using (1.3) on the result which completes the proof.

Considering that $T_N(q)$ is the coefficient of t^N in the sum

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(1-t)(tq^2;q^2)_n(tq;q^2)_n}$$

and observing again that $n^2+n = 2+4+\cdots+2n$ we see that the coefficient of t^N in

$$\frac{t^n q^{n^2+n}}{(tq^2;q^2)_n(tq;q^2)_n}$$

is the generating function for partitions into exactly N parts in which the largest part is even and every even smaller the largest part appears at least once. From the presence of the factor $(1-t)$ in the denominator we have proved the following theorem:

Theorem 3.2: $T_n(q)$ is the generating function for partitions into at most N parts in which the largest part is even and every even smaller than the largest appears at least once.

Replacing now q by 1 in (3.2) we get

$$T_0(1) = 1;\ T_1(1) = 2$$
$$T_n(1) = 3T_{n-1}(1) - T_{n-2}(1).$$

But for F_n we have

$$F_1 = 1;\ F_3 = 2$$
$$F_{2n+1} = 3F_{2n-1} - F_{2n-3}$$

which allow us to conclude that

$$B_n(1) = T_n(1) = F_{2n+1}$$

and by these results we have proved.

Theorem 3.3: The total number of partitions into at most N parts in which the largest part is even and every even smaller than the largest part appears at least once is equal to F_{2n+1}.

For family (3.2) we have also that, at $q=-1$, we get all the Fibonacci numbers F_n, $n \geq 2$.

$$T_0(-1) = 1;\ T_1(-1) = 2$$
$$T_n(-1) = T_{n-1}(-1) + T_{n-2}(-1)$$

i.e., $T_n(-1) = F_{n+2}, n \geq 0$.

If we make the same observation that have made for the first family of polynomials regarding the combinatorial interpretation at $q = -1$ we have proved the following result:

Theorem 3.4: The total number of partitions into at most N parts in which the largest part is even and every even smaller than the largest part appears at least once and having an even number of odd parts minus the number of those with an odd number of odd parts is equal to F_{N+2}.

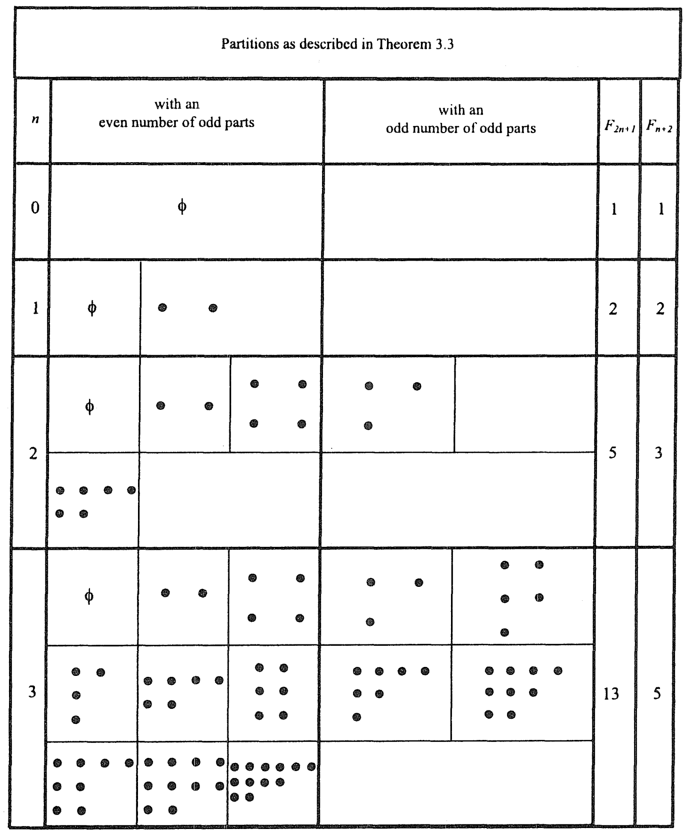

Table 3.1

In the table (3.1) we present, for a few values of n, all the results proved in this section. The first column has n, the second the partitions described in Theorem 3.3 with an even number of odd parts and the third column those with an odd number of odd parts. The fourth column has F_{2n+1} which is the total number of partitions in columns 2 and 3 and the fifth column has the difference between the number of partitions on the second and third column which is F_{n+2}.

4. A FORMULA FOR F_n

Using the fact that the Gaussian polynomials given in (1.6) are q-analogue of the binomial coefficient, i.e., that

$$\lim_{q\to 1}\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}$$

we may take the limits as q approaches 1 in (2.3) and (3.3) to get

$$\lim_{q\to 1} C_n(q) = \lim_{q\to 1}\left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j}\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3}\begin{bmatrix} 2n+1 \\ n-5j-2 \end{bmatrix}\right)$$

$$= \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2}\right] = C_n(1)$$

and

$$\lim_{q\to 1} B_n(q) = \lim_{q\to 1}\left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix}\right)$$

$$= \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1}\right] = B_n(1)$$

But as we have observed

$$C_n(1) = F_{2n+2} \text{ and } B_n(1) = F_{2n+1}$$

which tell us that

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2} \right] \tag{4.1}$$

and

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1} \right] \tag{4.2}$$

5. LATTICE PATHS AND FIBONACCI NUMBERS

In this section we are going to show how to express the Fibonacci numbers in terms of lattice path.

In Narayana [4], Lemma 4A one can find the following formula

$$|L(m,n;t,s)| \sum_{j=-\infty}^{\infty} \left[\binom{m+n}{m-k(t+s)} - \binom{m+n}{n+k(t+s)+t} \right] \tag{5.1}$$

which give the total number of lattice paths from the origin to (m,n) not touching the lines $y = x - t$ and $y = x + s$.

But considering that we can write (4.1) and (4.2) as follows

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{n+(n+1)}{n-j(2+3)} - \binom{n+(n+1)}{n+1+j(2+3)+2} \right] \tag{5.2}$$

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{n+(n+1)}{n-j(1+4)} - \binom{n+(n+1)}{(n+1)+j(1+4)+1} \right] \tag{5.3}$$

we can conclude just by comparing (4.4) and (4.5) with (4.3) that the following theorem holds:
Theorem 5.1: F_{2n+i} is the number of lattice paths from the origin to $(n, n+1)$ not touching the line $y = x - i$ and $y = x + 5 - i$, where $i = 1, 2$.

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AMS Classification Numbers: 11A17, 11P83, 11B39

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