# ELEMENTARY PROBLEMS AND SOLUTIONS

*Edited by*
**Russ Euler and Jawad Sadek**

*Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.*

*If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.*

*Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2004. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".*

## BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

## PROBLEMS PROPOSED IN THIS ISSUE

**B-961 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA**

Show that $\frac{L_{n+1}}{\alpha^{n-1}} + \frac{L_n}{\alpha^n}$ is a constant for all nonnegative integers $n$.

**B-962 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA**

Find

$$\prod_{k=1}^{\infty} \frac{F_{2k}F_{2k+2} + F_{2k-1}F_{2k+2}}{F_{2k}F_{2k+2} + F_{2k}F_{2k+1}}.$$

**B-963 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI**

Prove that

$$\frac{F_{2n+1}-1}{F_{2n+4}-3F_{n+2}-L_{n+2}+3} \geq \frac{1}{n}$$

for all $n \geq 1$.

**B-964 Proposed by Stanley Rabinowitz, MathPro, Westford, MA**

Find a recurrence relation for $r_n = \frac{F_n}{L_n}$.

**B-965 Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain**

Let $n$ be a positive integer. Prove that

$$\frac{F_n!(4F_{n+1})!}{(2F_n)!(F_{n-1}+F_{n+1})!(2F_{n+1})!}$$

is an integer.

## SOLUTIONS

### When Do they Converge?

**B-946 Proposed by Mario Catalani, University of Torino, Torino, Italy (Vol. 40, no. 5, November 2002)**

Find the smallest positive integer $k$ such that the following series converge and find the value of the sums:

$$1.\quad \sum_{i=1}^{\infty}\frac{i^2F_iL_i}{k^i} \qquad 2.\sum_{i=1}^{\infty}\frac{iF_i^2}{k^i}$$

**Solution by Toufik Mansour, Chalmers University of Technology, Sweden.**

1. Using Lemma 3.2 in [1], we get

$$G(x) = \sum_{n\geq 0} F_nL_nx^n = \frac{x}{x^2-3x+1}.$$

It follows that

$$\sum_{n\geq 0} n^2F_nL_nx^n = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}G(x)\right) = \frac{x(x^4+3x^3-6x^2+3x+1)}{(x^2-3x+1)^3}.$$

Hence, the sum $\sum_{n\geq 0} n^2 F_n L_n x^n$ converge if and only if $|x| < \frac{1}{2}(3-\sqrt{5})$, which means $k > \frac{1}{2}(3+\sqrt{5})$. Therefore, the smallest positive integer k such that (1) converges is $k=3$. In this case the sum (1) equals 354.

2. Using Corollary 3.5 in [1], we get

$$H(x) = \sum_{n\geq 0} F_n^2 x^n = \frac{x(1-x)}{(1+x)(1-3x+x^2)}.$$

It follows that

$$\sum_{n\geq 0} nF_n^2 x^n = x\frac{\partial}{\partial x}H(x) = \frac{(1-2x-2x^3+x^4+4x^2)x}{(1+x)^2(1-3x+x^2)^2}.$$

Hence, the sum $\sum_{n\geq 0} nF_n^2 x^n$ converges when $|x| < \frac{1}{2}(3-\sqrt{5})$, which means $k > \frac{1}{2}(3+\sqrt{5})$. Thererore, the smallest positive integer $k$ such that (2) converges is $k=3$. In this case, the sum (2) equals $\frac{87}{8}$. $\square$

P.S. It is easy to prove by induction that the sums $\sum_{i\geq 1} i^m F_i L_i x^i$ and $\sum_{i\geq 1} i^m F_i^2 x^i$ converge for all $x$ such that $|x| < \frac{1}{2}(3-\sqrt{5})$ (maximum domain), for all $m \geq 1$.

## REFERENCES

[1] P. Haukkanen. "A Note on Horadam's Sequence." *The Fibonacci Quarterly* **40.4** (20002): 358-361.

**Also solved by Paul Bruckman, Charles Cook, Kenneth Davenport, L.G. Dresel, Ovidiu Furdiu, Walther Janous, Harris Kwang, David Manes, James Sellers, and the proposer.**

## Integral and Nonsquare!

**B-947 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA (Vol. 40, no. 5, November 2002)**

(a) Find a nonsquare polynomial $f(x,y,z)$ with integer coefficients such that $f(F_n, F_{n+1}, F_{n+2})$ is a perfect square for all $n$.

(b) Find a nonsquare polynomial $g(x,y)$ with integer coefficients such that $g(F_n, F_{n+1})$ is a perfect square for all $n$.

**Solution by Paul Bruckman, Berkeley, CA and Walther Janous, Ursulinengymnasium, Innsbruck, Austria (separately).**

We begin with the well-known "Wronskian" identity:

$$F_{n+1}F_{n-1} - (F_n)^2 = (-1)^n \tag{1}$$

Two alternative forms of this identity are the following:

$$(F_{n+1})^2 - F_{n+1}F_n - (F_n)^2 = (-1)^n \tag{2}$$

$$(F_{n+1})^2 - F_{n+2}F_n = (-1)^n \tag{3}$$

This suggests the following solution for Part (a):

$$f(x,y,z) = (y^2 - xz)(y^2 - xy - x^2) = y^4 - xy^3 - x^2y^2 - xy^2z + x^2yz + x^3z \tag{4}$$

We see from (2) and (3) that with this $f$, we have: $f(F_n, F_{n+1}, F_{n+2}) = 1$, which is certainly a perfect square for all $n$.

Also, using (2), we may take the following solution for Part (b):

$$g(x,y) = (y^2 - xy - x^2 + 1)(y^2 - xy - x^2 - 1) = y^4 - 2xy^3 - x^2y^2 + 2x^3y + x^4 - 1 \tag{5}$$

It is easily checked that $g(F_n, F_{n+1}) = 0$ for all $n$, which is again a perfect square.
**Also solved by Peter Anderson, Michel Ballieu, L.G. Dresel, Ovidiu Furdui (part (a)), David Manes, and the proposer.**

## A Series Inequality

**B-948 Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain**
**(Vol. 40, no. 5, November 2002)**

Let $\ell$ be a positive integer greater than or equal to 2. Show that, for $x > 0$,

$$\log_{F_{\ell+1}F_{\ell+2}\dots F_{\ell+n}} x^{n^2} \leq \sum_{k=1}^{n} \log_{F_{\ell+k}} x.$$

**Remark.** The condition on $x$ should be $x > 1$. For example, the inequality fails when $n = \ell = 2$ (try, for instance, $x = 1/6$). The proof below shows why we need $x > 1$.

**Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY.**

It follows from Cauchy-Schwarz inequality that

$$n^2 = \left(\sum_{k=1}^{n} \sqrt{\ln F_{\ell+k}} \cdot \frac{1}{\sqrt{\ln F_{\ell+k}}}\right)^2 \leq \left(\sum_{k=1}^{n} \ln F_{\ell+k}\right)\left(\sum_{k=1}^{n} \frac{1}{\ln F_{\ell+k}}\right).$$

For $x > 1$, we have $\ln x > 0$. Hence

$$\frac{n^2 \ln x}{\ln F_{\ell+1}F_{\ell+2}\dots F_{\ell+n}} \leq \sum_{k=1}^{n} \frac{\ln x}{\ln F_{\ell+k}},$$

which completes the proof, because $\ln x/\ln a = \log_a x$ for any $a > 0$.

**Also solved by Paul Bruckman, Mario Catalani, L.G. Dresel, Ovidiu Furdui, Walther Janous, H.-J. Seiffert, and the proposer.**

## Couples Congruence

**B-949 Proposed by N. Gauthier, Royal Military College of Canada (Vol. 40, no. 5, November 2002)**

For $l$ and $n$ positive integers, find closed form expressions for the following sums,

$$S_1 \equiv \sum_{k=1}^{n} 3^{n-k} F^3_{3^k \cdot 2l} \text{ and } S_2 \equiv \sum_{k=1}^{n} 3^{n-k} L^3_{3^k \cdot (2l+1)}.$$

**Solution by Mario Catalini, University of Torino, Torino, Italy**

We will use the following identities:

$$5F_n^3 = F_{3n} - 3(-1)^n F_n, \tag{1}$$

$$L_n^3 = L_{3n} + 3(-1)^n L_n. \tag{2}$$

$$\sum_{k=1}^{n} 3^{n-k} F^3_{3^k \cdot 2l} = 3^{n-1} F^3_{3 \cdot 2l} + 3^{n-2} F^3_{3^2 \cdot 2l} + 3^{n-3} F^3_{3^3 \cdot 2l} + \cdots + 3F^3_{3^{n-1} \cdot 2l} + F^3_{3^n \cdot 2l}.$$

Using identity (1) and the fact that the subscript is always an even number we get

$$\begin{aligned} 5S_1 = 3^{n-1}[F_{3^2 \cdot 2l} - 3F_{3 \cdot 2l}] + 3^{n-2}[F_{3^3 \cdot 2l} - 3F_{3^2 \cdot 2l}] + 3^{n-3}[F_{3^4 \cdot 2l} - 3F_{3^3 \cdot 2l}] \\ + \cdots + 3[F_{3^n \cdot 2l} - 3F_{3^{n-1} \cdot 2l}] + [F_{3^{n+1} \cdot 2l} - 3F_{3^n \cdot 2l}]. \end{aligned}$$

Because of a telescopic effect we obtain simply

$$5S_1 = -3^n F_{3 \cdot 2l} + F_{3^{n+1} \cdot 2l}.$$

For the second summation we have

$$\sum_{k=1}^{n} 3^{n-k} L^3_{3^k \cdot (2l+1)} = 3^{n-1} L^3_{3 \cdot (2l+1)} + 3^{n-2} L^3_{3^2 \cdot (2l+1)} + 3^{n-3} L^3_{3^3 \cdot (2l+1)} + \cdots$$

$$+ 3L^3_{3^{n-1} \cdot (2l+1)} + L^3_{3^n \cdot (2l+1)}.$$

Using identity 2 and the fact that the subscript is always an odd number we get

$$S_2 = 3^{n-1}\left[L_{3^2\cdot(2l+1)} - 3L_{3\cdot(2l+1)}\right] + 3^{n-2}\left[L_{3^3\cdot(2l+1)} - 3L_{3^2\cdot(2l+1)}\right]$$

$$+ 3^{n-3}\left[L_{3^4\cdot(2l+1)} - 3L_{3^3\cdot(2l+1)}\right] + \dots$$

$$+ 3\left[L_{3^n\cdot(2l+1)} - 3L_{3^{n-1}\cdot(2l+1)}\right] + \left[L_{3^{n+1}\cdot(2l+1)} - 3L_{3^n\cdot(2l+1)}\right].$$

Because of a telescopic effect we obtain

$$S_2 = -3^n L_{3\cdot(2l+1)} + L_{3^{n+1}\cdot(2l+1)}.$$

**Also solved by Paul Bruckman, H.-J. Seiffert, and the proposer.**

## Primes $\cdots$ Again

**B-950 Proposed by Paul S. Bruckman, Berkeley, CA**
**(Vol. 40, no. 5, November 2002)**

For all primes $p > 2$, prove that

$$\sum_{k=1}^{p-1} \frac{F_k}{k} \equiv 0 \pmod{p},$$

where $\frac{1}{k}$ represents the residue $k^{-1} (\text{mod } p)$.

H.J. Seiffert refers the reader to part (b) of problem H-545 in *The Fibonacci Quarterly* **38.2** (2000): 187-188 and Kenneth B. Davenport quotes Corollary 4 of "Equivalent Conditions for Fibonacci and Lucas Pseudoprimes which Contain a Square Factor," *Pi Mu Epsilon Journal* **10.8** Spring 1988, 634-642.

**Also solved by L.G. Dresel and the proposer.**