ON POSITIVE NUMBERS n FOR WHICH $\Omega(n)$ DIVIDES F_n

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(Submitted June 4, 2001)

1. INTRODUCTION

Let n be a positive integer n and let $\omega(n)$, $\Omega(n), \tau(n)$, $\phi(n)$ and $\sigma(n)$ be the classical arithmetic functions of n. That is, $\omega(n)$, $\Omega(n)$, and $\tau(n)$ count the number of distinct prime divisors of n, the total number of prime divisors of n, and the number of divisors of n, respectively, while $\phi(n)$ and $\sigma(n)$ are the Euler function of n and the sum of divisors function of n respectively.

A lot of interest has been expressed in investigating the asymptotic densities of the sets of n for which one of the "small" arithmetic functions of n divides some other arithmetic function of n. For example, in [2], it was shown that the set of n for which $\omega(n)$ divides n is of asymptotic density zero. This result was generalized in [4]. The formalism from [4] implies, in particular, that the set of n for which either $\Omega(n)$ or $\tau(n)$ divide n is also of asymptotic density zero. On the other hand, in [1] it is shown that $\tau(n)$ divides $\sigma(n)$ for almost all n and, in fact, it can be shown that all three numbers $\omega(n)$, $\Omega(n)$ and $\tau(n)$ divide both $\phi(n)$ and $\sigma(n)$ for almost all n.

In this note, we look at the set of positive integers n for which one of the small arithmetic functions of n divides F_n or L_n. Here, F_n and L_n are the n^{th} Fibonacci numbers and the n^{th} Lucas number, respectively. We have the following result:

Theorem:

The set of n for which either one of the numbers $\omega(n)$, $\Omega(n)$ or $\tau(n)$ divides F_{2n} is of asymptotic density zero.

Since $F_{2n} = F_n L_n$ for all $n \geq 0$, it follows that for most n, none of the numbers $\omega(n)$, $\Omega(n)$ or $\tau(n)$ divides either F_n or L_n. Following our method of proof, we can easily generalize the above Theorem to the case when the Fibonacci sequence is replaced by any Lucas or Lehmer sequence. We believe that the above Theorem should hold with the Fibonacci sequence replaced by any non-degenerate linearly recurrent sequence but we have not worked out the details of this statement.

2. PRELIMINARY RESULTS

Throughout the proof, we denote by $c_1, c_2, \dots$ computable constants which are absolute. For a positive integer k and a large positive real number x we let $\log_k(x)$ to be the composition of the natural logarithm with itself k times evaluated in x. Finally, assume that $\delta(x)$ is any function defined for large positive values of x which tends to infinity with x. We use p to denote a prime number. We begin by pointing out a "large" asymptotic set of positive integers n.

Lemma 1:

Let x be a large real number and let $A(x)$ be the set of all positive integers n satisfying the following conditions:

1. $\sqrt{x} < n < x$;
2. $|\omega(n) - \log_2(x)| < \delta(x)(\log_2(x))^{1/2}$ and $|\Omega(n) - \log_2(x)| < \delta(x)(\log_2(x))^{1/2}$;
3. *Write* $n = \prod_{p|n} p^{\alpha_p}$. *Then,* $\max_{p|n}(\alpha_p) < \log_3(x)$ *and if* $p > \log_3(x)$, *then* $\alpha_p = 1$.

Then $A(x)$ contains all positive integers $n < x$ except for $o(x)$ of them.

The Proof of Lemma 1:

1. Clearly, there are at most $\sqrt{x} = o(x)$ positive integers which do not satisfy 1.
2. By a result of Túran (see [6])

$$\sum_{n<x}(\omega(n) - \log_2(x))^2 = O(x\log_2(x)). \tag{1}$$

Thus, the inequality

$$|\omega(n) - \log_2(x)| < \frac{1}{2}\delta(x)(\log_2(x))^{1/2} \tag{2}$$

holds for all $n < x$ except for $O(\frac{x}{\delta(x)}) = o(x)$ of them. This takes care of the first inequality asserted at 2. For the second inequality here, we use the fact

$$\sum_{n<x}(\Omega(n) - \omega(n)) = O(x). \tag{3}$$

By (3), it follows that the inequality

$$\Omega(n) - \omega(n) < \frac{1}{2}\delta(x)(\log_2(x))^{1/2} \tag{4}$$

holds for all $n < x$ except for $O(\frac{x}{\delta(x)(\log_2(x))^{1/2}}) = o(x)$ of them. Inequalities (2) and (4) now tell us that

$$|\Omega(n) - \log_2(x)| < \delta(x)(\log_2(x))^{1/2} \tag{5}$$

holds for all $n < x$ except for $o(x)$ of them.

3. Assume first that n is divisible by some prime power p^α with $\alpha \geq \log_3(x)$. Then, the number of such $n < x$ is certainly at most

$$\sum_p \frac{x}{p^{\log_3(x)}} < x(\zeta(\log_3(x)) - 1) = O\left(\frac{x}{2^{\log_3(x)}}\right) = o(x). \tag{6}$$

Here, we used ζ to denote the classical Riemann zeta function.

Finally, assume that n is divisible by a square of a prime $p > \log_3(x)$. Then, the number of such $n < x$ is at most

$$\sum_{\log_3(x)<p} \frac{x}{p^2} = O\left(\frac{x}{\log_3(x)\log_4(x)}\right) = o(x). \tag{7}$$

Thus, $A(x)$ contains all positive integers $n < x$ but for $o(x)$ of them.

In what follows, for a positive integer n we denote by $z(n)$ the *order of apparition of* n in the Fibonacci sequence; that is, $z(n)$ is the smallest positive integer n for which $n|F_{z(n)}$. In the next Lemma, we recall a few well-known facts about $z(n)$.

Lemma 2:

1. *There exist two constants* c_1 *and* c_2 *such that*

$$c_1 \log n < z(n) < c_2 n \log_2(n) \quad \text{for all } n \geq 3. \tag{8}$$

2. $z(2^s) = 3 \cdot 2^{s-2}$*for all* $s \geq 3$.

The Proof of Lemma 2:

1. Let $\gamma_1 = \frac{1+\sqrt{5}}{2}$ be the golden section and let $\gamma_2 = \frac{1-\sqrt{5}}{2}$ be its conjugate. Since

$$F_n = \frac{\gamma_1^n - \gamma_2^n}{\gamma_1 - \gamma_2} \quad \text{for all } n \geq 0 \tag{9}$$

it follows easily that

$$F_n < \gamma_1^n \tag{10}$$

holds for all $n \geq 0$. Hence, since $n|F_{z(n)}$, we get, in particular, that

$$n \leq F_{z(n)} < \gamma_1^{z(n)}. \tag{11}$$

Taking logarithms in (11) we get

$$c_1 \log n < z(n) \tag{12}$$

with $c_1 = \frac{1}{\log \gamma_1}$.

For the upper bound for $z(n)$, we recall that if

$$n = \prod_{p|n} p^{\alpha_p} \tag{13}$$

then,

$$z(n) = \mathrm{lcm}_{p|n}(z(p^{\alpha_p})). \tag{14}$$

Moreover, if p is a prime, then

$$z(p)|p - \delta_p, \tag{15}$$

where $\delta_p = \left(\frac{p}{5}\right)$ is the Jacobi symbol of p in respect to 5, and if $\alpha \geq 2$ is a positive integer, then

$$z(p^\alpha)|p^{\alpha-1}z(p). \tag{16}$$

Combining (14), (15) and (16), we get that

$$z(n) \le \prod_{p|n} p^{\alpha_p - 1}(p+1) \le \sigma(n) < c_2 n \log_2(n). \tag{17}$$

2. This is well-known (see, for example, [5]).

For a given positive integer j and a positive large real number x let

$$\rho_j(x) = \#\{n < x | \omega(n) = j\} \tag{18}$$

and

$$\pi_j(x) = \#\{n < x | \Omega(n) = j\}. \tag{19}$$

We shall need the following result:

Lemma 3:

There exist two absolute constants c_3 and c_4 such that if $x > c_3$ and j is any positive integer, then

$$\max(\rho_j(x), \pi_j(x)) < \frac{c_4 x}{(\log_2(x))^{1/2}}. \tag{20}$$

The Proof of Lemma 3: This is well-known (see [3], page 303).

We are now ready to prove the Theorem.

3. THE PROOF OF THE THEOREM

We assume that x is large and that $n \in A(x)$, where $A(x)$ is the set defined in Lemma 1 for some function δ.

Throughout the proof, we assume that $\delta(x)$ is any function tending to infinity with x slower than $(\log_2(x))^{1/2}$; that is

$$\lim_{x \to \infty} \frac{\delta(x)}{(\log_2(x))^{1/2}} = 0. \tag{21}$$

We first treat the easiest case, namely $\tau(n)|F_{2n}$. Since $n \in A(x)$, it follows that

$$\omega(n) > \frac{1}{2} \log_2(x) \tag{22}$$

holds for x large enough. Now write

$$n_1 = \prod_{p|n,\ p<\log_3(x)} p^{\alpha_p}$$

and

$$n_2 = \prod_{p|n,\ p\ge\log_3(x)} p.$$

Clearly, $n = n_1 n_2, n_1$ and n_2 and coprime and n_2 is square-free, therefore

$$\omega(n_2) = \omega(n) - \omega(n_1) > \frac{1}{2}\log_2(x) - \pi(\log_3(x)) > \frac{1}{3}\log_2(x), \tag{23}$$

where the last inequality in (23) holds for x large enough. Now notice that

$$\tau(n) = \tau(n_1)\tau(n_2) = 2^{\omega(n_2)}\tau(n_1). \tag{24}$$

Thus, if $\tau(n)|F_{2n}$, we get, in particular, that $2^{\omega(n_2)}|F_{2n}$, whence $z(2^{\omega(n_2)})|2n$. By inequality (23) and Lemma 2, it follows that if we denote by α_2 the exponent at which 2 divides n, then

$$\alpha_2 > \omega(n_2) - 3 > \frac{1}{3}\log_2(x) - 3. \tag{25}$$

The expression appearing in the right hand side of inequality (25) is larger than $\log_3(x)$ for large x, contradicting the fact that $n \in A(x)$. Thus, if x is large and $n \in A(x)$, then $\tau(n)$ cannot divide F_{2n}.

We now treat the cases in which $\omega(n)$ or $\Omega(n)$ divides F_{2n}. As the reader will see, the key ingredients for these proofs are the fact that n satisfies both condition 2 of Lemma 1 as well as Lemma 3, and both these results are symmetric in $\omega(n)$ and $\Omega(n)$. Thus, we shall treat in detail only the case in which $\omega(n)$ divides F_{2n}.

We fix a positive integer j such that

$$|j - \log_2(x)| < \delta(x)(\log_2(x))^{1/2} \tag{26}$$

and we find an upper bound for the set of $n \in A(x)$ for which $\omega(n) = j$ and $j|F_{2n}$. Since $j|F_{2n}$, it follows that

$$2n = z(j)m, \tag{27}$$

for some positive integer m. Assume first that n is odd. In this case,

$$j + 1 = \omega(2n) = \omega(mz(j)) = \omega(m) + \omega(z(j)) - s, \quad \text{where } s = \omega(\gcd(m, z(j))). \tag{28}$$

We now notice that by inequality (26) and Lemma 2,

$$c_5 \log_3(x) < z(j) < c_6 \log_2(x)\log_4(x) \tag{29}$$

holds for all x large enough and uniformly in j. In particular,

$$s \le \omega(z(j)) < c_7 \log(z(j)) < c_8 \log_3(x) \tag{30}$$

holds for x large enough and uniformly in j. Assume that s is a fixed number in the set $\{0, 1, \ldots, \omega(z(j))\}$. Then

$$m = \frac{2n}{z(j)} < \frac{2x}{z(j)} \tag{31}$$

is a number with the property that

$$\omega(m) = j + 1 - \omega(z(j)) + s \tag{32}$$

is fixed. Moreover, it is easy to see that

$$x^{1/2} < \frac{2x}{c_6 \log_2(x) \log_4(x)} < \frac{2x}{z(j)} < \frac{2x}{c_5 \log_3(x)} < x. \tag{33}$$

Now Lemma 3 together with inequality (33) implies that the number of numbers $m < \frac{2x}{z(j)}$ for which $\omega(m)$ is given by formula (32) for fixed j and s is at most

$$\frac{c_9 x}{z(j)(\log_2(x))^{1/2}} \tag{34}$$

and this bound is uniform in j and s when x is large. We now let s vary and we get that the number of odd $n \in A(x)$ for which $\omega(n) = j$ and $j|F_{2n}$ is bounded above by

$$\frac{c_9(x(\omega(z(j)) + 1)}{z(j)(\log_2(x))^{1/2}} < \frac{c_{10} x \log(z(j))}{z(j)(\log_2(x))^{1/2}}. \tag{35}$$

We now use inequality (29) to conclude that (35) is bounded above by

$$\frac{c_{11} x \log_4(x)}{\log_3(x)(\log_2(x))^{1/2}}. \tag{36}$$

A similar analysis can be done to count the number of even $n \in A(x)$ for which $\omega(n) = j$ and $j|F_{2n}$. Thus, the total number of $n \in A(x)$ for which $\omega(n) = j$ and $j|F_{2n}$ is bounded above by

$$\frac{c_{12} x \log_4(x)}{\log_3(x)(\log_2(x))^{1/2}} \tag{37}$$

for large x and uniformly in j. Since $j = \omega(n)$ satisfies (26), it follows that j can take at most $2\delta(x)(\log_2(x))^{1/2} + 1$ values. Thus, the totality of $n \in A(x)$ for which $\omega(n)|F_{2n}$ is certainly not more than

$$\frac{c_{13} x \log_4(x) \delta(x)}{\log_3(x)}. \tag{38}$$

It now suffices to observe that one can choose $\delta(x)$ such that the function appearing at (38) is $o(x)$. For example, one can choose $\delta(x) = \frac{\log_3(x)}{\log_4(x)^2}$ and then the last expression appearing in (38) is $O(\frac{x}{\log_4(x)}) = o(x)$.

This shows that the set of n for which $\omega(n)|F_{2n}$ is of asymptotic density zero. As we mentioned before, a similar analysis can be done to treat the case in which $\Omega(n)|F_{2n}$. The Theorem is therefore proved.

4. REMARKS

One may ask what about the set of positive integers n for which one of the "large" arithmetic functions of n, i.e. $\phi(n)$ or $\sigma(n)$ divides F_n or L_n. The answer is that the sets of these n have all asymptotic densities zero, and this follows easily from our Theorem combined with the fact that both $\phi(n)$ and $\sigma(n)$ are divisible by all three numbers $\omega(n)$, $\Omega(n)$ and $\tau(n)$

for almost all n. If instead of considering whether or not F_n is a multiple of some other function of n, one looks at $F_{\phi(n)}$ or $F_{\sigma(n)}$, then one can show that both $F_{\phi(n)}$ and $F_{\sigma(n)}$ are divisible by all three numbers $\omega(n)$, $\Omega(n)$, $\tau(n)$ for almost all n. We do not give more details.

ACKNOWLEDGMENTS

I would like to thank Carl Pomerance for helpful advice and for suggestions which have improved the quality of this manuscript.

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AMS Classification Numbers: 11K65, 11N25

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