GENERATING FUNCTIONS, WEIGHTED AND NON-WEIGHTED SUMS FOR POWERS OF SECOND-ORDER RECURRENCE SEQUENCES

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1. INTRODUCTION

DeMoivre (1718) used the generating function (found by employing the recurrence) for the Fibonacci sequence $\sum_{i=0}^{\infty} F_i x^i = \frac{x}{1-x-x^2}$, to obtain the identities $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, L_n = \alpha^n + \beta^n$ (*Lucas numbers*) with $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$. These identities are called *Binet formulas*, in honor of Binet who in fact rediscovered them more than one hundred years later, in 1843 (see [6]). Reciprocally, using the Binet formulas, we can find the generating function easily $\sum_{i=0}^{\infty} F_i x^i = \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} (\alpha^i - \beta^i) x^i = \frac{1}{\sqrt{5}} \left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \right) = \frac{x}{1-x-x^2}$, since $\alpha\beta = -1, \alpha + \beta = 1$.

A natural question is whether we can find a closed form for the generating function for powers of Fibonacci numbers, or better yet, for powers of any second-order recurrence sequences. Carlitz [1] and Riordan [4] were unable to find the closed form for the generating functions $F(r,x)$ of F_n^r, but found a recurrence relation among them, namely

$$(1 - L_r x + (-1)^r x^2) F(r,x) = 1 + rx \sum_{j=1}^{[\frac{r}{2}]} (-1)^j \frac{A_{rj}}{j} F(r-2j, (-1)^j x),$$

with A_{rj} having a complicated structure (see also [2]). We are able to complete the study started by them by finding a closed form for the generating function for powers of any non-degenerate second-order recurrence sequence. We would like to point out, that this "forgotten" technique we employ can be used to attack successfully other sums or series involving any second-order recurrence sequence. We also find closed forms for non-weighted partial sums for nondegenerate second-order recurrence sequences, generalizing a theorem of Horadam [3] and also weighted (by the binomial coefficients) partial sums for such sequences. Using these results we indicate how to obtain some congruences modulo powers of 5 for expressions involving Fibonacci and/or Lucas numbers.

2. GENERATING FUNCTIONS

We consider the general nondegenerate second-order recurrence, $U_{n+1} = aU_n + bU_{n-1}, a, b, U_0, U_1$ integers, $\delta = a^2 + 4b \neq 0$. We intend to find the generating function of

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powers of its terms, $U(r,x)=\sum_{i=0}^{\infty}U_i^r x^i$. It is known that the Binet formula for the sequence U_n is $U_n=A\alpha^n-B\beta^n$, where $\alpha=\frac{1}{2}(a+\sqrt{a^2+4b}), \beta=\frac{1}{2}(a-\sqrt{a^2+4b})$ and $A=\frac{U_1-U_0\beta}{\alpha-\beta}, B=\frac{U_1-U_0\alpha}{\alpha-\beta}$. We associate the sequence $V_n=\alpha^n+\beta^n$, which satisfies the same recurrence, with the initial conditions $V_0=2, V_1=a$.

Theorem 1: *We have*

$$U(r,x)=\sum_{k=0}^{\frac{r-1}{2}}(-AB)^k\binom{r}{k}\frac{A^{r-2k}-B^{r-2k}+(-b)^k(B^{r-2k}\alpha^{r-2k}-A^{r-2k}\beta^{r-2k})x}{1-(-b)^kV_{r-2k}x-b^rx^2},$$

if r is odd, and

$$U(r,x)=\sum_{k=0}^{\frac{r}{2}-1}(-AB)^k\binom{r}{k}\frac{B^{r-2k}+A^{r-2k}-(-b)^k(B^{r-2k}\alpha^{r-2k}+A^{r-2k}\beta^{r-2k})x}{1-(-b)^kV_{r-2k}x+b^rx^2}$$

$$+\binom{r}{\frac{r}{2}}\frac{(-AB)^{\frac{r}{2}}}{1-(-b)^{\frac{r}{2}}x},\ \textit{if } r \textit{ is even.}$$

Proof: We evalute

$$U(r,x)=\sum_{i=0}^{\infty}\left(\sum_{k=0}^{r}\binom{r}{k}(A\alpha^i)^k(-B\beta^i)^{r-k}\right)x^i$$

$$=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}\sum_{i=0}^{\infty}(\alpha^k\beta^{r-k}x)^i$$

$$=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}\frac{1}{1-\alpha^k\beta^{r-k}x}.$$

If r is odd, then associating $k\leftrightarrow r-k$, we get

$$U(r,x)=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k}{1-\alpha^{r-k}\beta^kx}-\frac{A^kB^{r-k}}{1-\alpha^k\beta^{r-k}x}\right)$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\frac{A^{r-k}B^k-A^kB^{r-k}+(A^kB^{r-k}\alpha^{r-k}\beta^k-A^{r-k}B^k\alpha^k\beta^{r-k})x}{1-(\alpha^k\beta^{r-k}+\alpha^{r-k}\beta^k)x+\alpha^r\beta^rx^2}$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}.\frac{A^{r-k}B^k-A^kB^{r-k}+(-b)^k(A^kB^{r-k}\alpha^{r-2k}-A^{r-k}B^k\beta^{r-2k})x}{1-(-b)^kV_{r-2k}x-b^rx^2}$$

If r is even, then associating $k \leftrightarrow r-k$, except for the middle term, we get

$$U(r,x) = \sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\left(\frac{A^kB^{r-k}}{1-\alpha^k\beta^{r-k}x}+\frac{A^{r-k}B^k}{1-\alpha^{r-k}\beta^kx}\right)+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-b)^{\frac{r}{2}}x}$$

$$= \sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\frac{A^kB^{r-k}+A^{r-k}B^k-(A^kB^{r-k}\alpha^{r-k}\beta^k+A^{r-k}B^k\alpha^k\beta^{r-k})x}{1-(\alpha^k\beta^{r-k}+\alpha^{r-k}\beta^k)x+\alpha^r\beta^rx^2}$$

$$+\binom{r}{\frac{r}{2}}\frac{(-AB)^{\frac{r}{2}}}{1-(-b)^{\frac{r}{2}}x}$$

$$= \sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\frac{A^kB^{r-k}+A^{r-k}B^k-(-b)^k(A^kB^{r-k}\alpha^{r-2k}+A^{r-k}B^k\beta^{r-2k})x}{1-(-b)^kV_{r-2k}x+b^rx^2}$$

$$+\binom{r}{\frac{r}{2}}\frac{(-AB)^{\frac{r}{2}}}{1-(-b)^{\frac{r}{2}}x}.$$

If $U_0=0$, then $A=B=\frac{U_1}{\alpha-\beta}$, and in this case we can derive the following beautiful identities.

Theorem 2: *We have*

$$U(r,x) = A^{r-1}\sum_{k=0}^{\frac{r-1}{2}}\binom{r}{k}\frac{b^kU_{r-2k}x}{1-(-b)^kV_{r-2k}x-b^rx^2},\ \textit{if } r \textit{ is odd}$$

$$U(r,x) = A^r\sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\frac{2-(-b)^kV_{r-2k}x}{1-(-b)^kV_{r-2k}x+b^rx^2}+\binom{r}{\frac{r}{2}}\frac{(-1)^{\frac{r}{2}}A^r}{1-(-b)^{\frac{r}{2}}x},\ \textit{if } r \textit{ is even.}$$

Corollary 3: *If $\{U_n\}_n$ is a nondegenerate second-order recurrence sequence and $U_0=0$, then*

$$U(1,x) = \frac{U_1x}{1-ax-bx^2} \tag{1}$$

$$U(2,x) = \frac{U_1^2x(1-bx)}{(bx+1)(b^2x^2-V_2x+1)} \tag{2}$$

$$U(3,x) = \frac{\delta A^2U_1x(1-2abx-b^3x^2)}{(1-V_3x-b^3x^2)(1+bV_1x-b^3x^2)}. \tag{3}$$

Proof: We use Theorem 2. The first two identities are straightforward. Now,

$$\begin{aligned} U(3,x) &= A^2\left(\frac{U_3x}{1-V_3x-b^3x^2}+\binom{3}{1}\frac{bU_1x}{1+bV_1x-b^3x^2}\right) \\ &= A^2x\frac{U_3+3bU_1+b(U_3V_1-3U_1V_3)x-b^3(U_3+3bU_1)x^2}{(1-V_3x-b^3x^2)(1+bV_1x-b^3x^2)} \\ &= \frac{\delta A^2U_1x(1-2abx-b^3x^2)}{(1-V_3x-b^3x^2)(1+bV_1x-b^3x^2)}, \end{aligned}$$

since $U_3+3bU_1=(a^2+4b)U_1=\delta U_1$ and $U_3V_1-3U_1V_3=-2a\delta U_1$. □

Remark 4: *If $U_n=F_n$, the Fibonacci sequence, then $a=b=1$, and if $U_n=P_n$, the Pell sequence, then $a=2, b=1$.*

3. HORADAM'S THEOREM

Horadam [3] found some closed forms for partial sums $S_n=\sum_{i=1}^n P_i$, $S_{-n}=\sum_{i=1}^n P_{-i}$, where P_n is the generalized Pell sequence, $P_{n+1}=2P_n+P_{n-1}$, $P_1=p$, $P_2=q$. Let p_n be the ordinary Pell sequence, with $p=1, q=2$, and q_n be the sequence satisfying the same recurrence, with $p=1, q=3$. He proved

Theorem 5 (Horadam): *For any n,*

$$\begin{aligned} S_{4n} &= q_{2n}(pq_{2n-1}+qq_{2n})+p-q; & S_{4n-2} &= q_{2n-1}(pq_{2n-2}+qq_{2n-1}) \\ S_{4n+1} &= q_{2n}(pq_{2n}+qq_{2n+1})-q; & S_{4n-1} &= q_{2n}(pq_{2n-2}+qq_{2n-1})-p \\ S_{-4n} &= q_{2n}(-pq_{2n+2}+qq_{2n+1})+3p-q; & S_{-4n+2} &= q_{2n}(-pq_{2n}+qq_{2n-1})+2p \\ S_{-4n+1} &= q_{2n}(pq_{2n+1}-qq_{2n})+p; & S_{-4n-1} &= q_{2n+1}(pq_{2n+2}-qq_{2n+1})+2p-q. \end{aligned}$$

We observe that Horadam's theorem is a particular case of the partial sum for a non-degenerate second-order recurrence sequence U_n. In fact, we generalize it even more by finding $S_{n,r}^U(x)=\sum_{i=0}^n U_i^r x^i$. For simplicity, we let $U_0=0$. Thus, $U_n=A(\alpha^n-\beta^n)$ and $V_n=\alpha^n+\beta^n$. We prove

Theorem 6: *We have*

$$S_{n,r}^U(x)=A^{r-1}x\sum_{k=0}^{\frac{r-1}{2}}b^k\binom{r}{k}\frac{U_{r-2k}-(-b)^{kn}U_{(r-2k)(n+1)}x^n+(-b)^{r+k(n-1)}U_{(r-2k)n}x^{n+1}}{1-(-b)^kV_{r-2k}x-b^rx^2}, \quad (4)$$

if r is odd, and

$$S^U_{n,r}(x) = A^r(-1)^{\frac{r}{2}}\binom{r}{\frac{r}{2}}\frac{(-b)^{\frac{r}{2}(n+1)}x^{n+1}-1}{(-b)^{\frac{r}{2}}x-1} + A^r\sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\cdot$$

$$\frac{2-(-b)^kV_{r-2k}x-(-b)^{k(n+1)}V_{(r-2k)(n+1)}x^{n+1}+(-b)^{r+kn}V_{(r-2k)n}x^{n+2}}{1-(-b)^kV_{r-2k}x+b^rx^2}, \tag{5}$$

if r is even.

Proof: We evaluate

$$\begin{aligned} S^U_{n,r}(x) &= \sum_{i=0}^{n}\sum_{k=0}^{r}\binom{r}{k}(A\alpha^i)^k(-A\beta^i)^{r-k}x^i \\ &= A^r\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\sum_{i=0}^{n}(\alpha^k\beta^{r-k}x)^i \\ &= A^r\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\frac{(\alpha^k\beta^{r-k}x)^{n+1}-1}{\alpha^k\beta^{r-k}x-1}. \end{aligned}$$

Assume r is odd. Then, associating $k \leftrightarrow r-k$, we get

$$S_{n,r}^{U}(x) = A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \left(\frac{(\alpha^{r-k}\beta^k x)^{n+1} - 1}{\alpha^{r-k}\beta^k x - 1} - \frac{(\alpha^k \beta^{r-k} x)^{n+1} - 1}{\alpha^k \beta^{r-k} x - 1} \right)$$

$$= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{(\alpha^k \beta^{r-k} x - 1)(\alpha^{(r-k)(n+1)} \beta^{k(n+1)} x^{n+1} - 1)}{}$$

$$\frac{-(\alpha^{r-k}\beta^k x - 1)(\alpha^{k(n+1)}\beta^{(r-k)(n+1)} x^{n+1} - 1)}{(\alpha^k \beta^{r-k} x - 1)(\alpha^{r-k}\beta^k x - 1)}$$

$$= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{(\alpha^{r(n+1)-kn}\beta^{r+kn} x^{n+2} - \alpha^{(r-k)(n+1)}\beta^{k(n+1)} x^{n+1}}{}$$

$$\frac{-\alpha^k \beta^{r-k} x - \alpha^{r+kn}\beta^{r(n+1)-kn} x^{n+2} + \alpha^{r-k}\beta^k x}{}$$

$$\frac{+\alpha^{k(n+1)}\beta^{(r-k)(n+1)} x^{n+1}}{1 - (-b)^k(\alpha^{r-2k} + \beta^{r-2k})x + \alpha^r \beta^r x^2}$$

$$= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{(-b)^k(\alpha^{r-2k} - \beta^{r-2k})x - (-b)^{k(n+1)}(\alpha^{(r-2k)(n+1)}}{}$$

$$\frac{-\beta^{(r-2k)(n+1)})x^{n+1} + (-b)^{r+kn}(\alpha^{(r-2k)n} - \beta^{(r-2k)n})x^{n+2}}{1 - (-b)^k V_{r-2k} x - b^r x^2}$$

$$= A^{r-1} x \sum_{k=0}^{\frac{r-1}{2}} b^k \binom{r}{k} \frac{U_{r-2k} - (-b)^{kn} U_{(r-2k)(n+1)} x^n + (-b)^{r+k(n-1)} U_{(r-2k)n} x^{n+1}}{1 - (-b)^k V_{r-2k} x - b^r x^2}.$$

Assume r is even. Then, as before, associating $k \leftrightarrow r-k$, except for the middle term, we get

$$S_{n,r}^{U}(x) = A^r \sum_{k=0}^{\frac{r}{2}-1} (-1)^k \binom{r}{k} \frac{2-(-b)^k(\alpha^{r-2k}+\beta^{r-2k})x-(-b)^{k(n+1)}(\alpha^{(r-2k)(n+1)}}{}$$

$$\frac{+\beta^{(r-2k)(n+1)})x^{n+1}+(-b)^{r+kn}(\alpha^{(r-2k)n}+\beta^{(r-2k)n})x^{n+2}}{1-(-b)^kV_{r-2k}x+b^rx^2}$$

$$+A^r(-1)^{\frac{r}{2}}\binom{r}{\frac{r}{2}}\frac{(-b)^{\frac{r}{2}(n+1)}x^{n+1}-1}{(-b)^{\frac{r}{2}}x-1}$$

$$=A^r(-1)^{\frac{r}{2}}\binom{r}{\frac{r}{2}}\frac{(-b)^{\frac{r}{2}(n+1)}x^{n+1}-1}{(-b)^{\frac{r}{2}}x-1}+A^r\sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\cdot$$

$$\frac{2-(-b)^kV_{r-2k}x-(-b)^{k(n+1)}V_{(r-2k)(n+1)}x^{n+1}+(-b)^{r+kn}V_{(r-2k)n}x^{n+2}}{1-(-b)^kV_{r-2k}x+b^rx^2}. \quad \square$$

Taking $r=1$, we get the partial sum for any nondegenerate second-order recurrence sequence, with $U_0=0$,

Corollary 7: $S_{n,1}^{U}(x)=\frac{x(U_1-U_{n+1}x^n-bU_nx^{n+1})}{1-V_1x-bx^2}$

Remark 8: *Horadam's theorem follows easily, since* $S_n = S_{n,1}^{P}(1)$. *Also* S_{-n} *can be found without difficulty, by observing that* $P_{-n}=pp_{-n-2}+qp_{-n-1}=-p(-1)^{n+2}p_{n+2}-q(-1)^{n+1}p_{n+1}$, *and using* $S_{n,1}^{p}(-1)$.

4. WEIGHTED COMBINATORIAL SUMS

In [6] there are quite a few identities like $\sum_{i=0}^{n}\binom{n}{i}F_i=F_{2n}$, or $\sum_{i=0}^{n}\binom{n}{i}F_i^2$, which is $5^{[\frac{n-1}{2}]}L_n$ if n is even, and $5^{[\frac{n-1}{2}]}F_n$, if n is odd. A natural question is: *for fixed* r, *what is the closed form for the weighted sum* $\sum_{i=0}^{n}\binom{n}{i}F_i^r$ *(if it exists)?* We are able to answer the previous question, not only for the Fibonacci sequence, but also for any second-order recurrence sequence U_n, in a more general setting. Let $S_{r,n}(x)=\sum_{i=0}^{n}\binom{n}{i}U_i^rx^i$.

Theorem 9: *We have*

$$S_{r,n}(x)=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}(1+\alpha^k\beta^{r-k}x)^n.$$

Moreover, if $U_0=0$, *then* $S_{r,n}(x)=A^r\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}(1+\alpha^k\beta^{r-k}x)^n$.

Proof: Let

$$S_{r,n}(x) = \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{r} \binom{r}{k} (A\alpha^i)^k (-B\beta^i)^{r-k} x^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} A^k (-B)^{r-k} \sum_{i=0}^{n} \binom{n}{i} (\alpha^k \beta^{r-k} x)^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} A^k (-B)^{r-k} (1 + \alpha^k \beta^{r-k} x)^n$$

If $U_0 = 0$, then $A = B$, and $S_{r,n}(x) = A^r \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} (1 + \alpha^k \beta^{r-k} x)^n$. □

Although we found an answer, it is not very exciting. However, by studying Theorem 9, we observe that we might be able to get nice sums involving the Fibonacci and Lucas sequences (or any such sequence, for that matter), if we are able to express 1 plus/minus a power of α, β as *the same* multiple of a power of α, respectively β. When $U_n = F_n$, the Fibonacci sequence, the following lemma does exactly what we need.

Lemma 10: *The following identities are true*

$$\begin{aligned} \alpha^{2s} - (-1)^s &= \sqrt{5}\alpha^s F_s \\ \beta^{2s} - (-1)^s &= -\sqrt{5}\beta^s F_s \\ \alpha^{2s} + (-1)^s &= L_s \alpha^s \\ \beta^{2s} + (-1)^s &= L_s \beta^s. \end{aligned} \tag{6}$$

Proof: Straightforward using the Binet formula for F_s and L_s. □

Theorem 11: *We have*

$$S_{4r+2,n}(1) = 5^{\frac{n+1}{2} - (2r+1)} \sum_{k=0}^{2r} \binom{4r+2}{k} F_{2r+1-k}^n F_{n(2r+1-k)}, \textit{ if } n \textit{ is odd} \tag{7}$$

$$S_{4r+2,n}(1) = 5^{\frac{n}{2} - (2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} F_{2r+1-k}^n L_{n(2r+1-k)}, \textit{ if } n \textit{ is even} \tag{8}$$

$$S_{4r,n}(1) = 5^{-2r} \left[\sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} L_{2r-k}^n L_{(2r-k)n} + 2^n \binom{4r}{2r} \right]. \tag{9}$$

Proof: We use Theorem 9. Associating $k \leftrightarrow 4r+2-k$, except for the middle term in $S_{4r+2,n}(1)$, we obtain

$$S_{4r+2,n}(1) = 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left[\left(1+\alpha^k \beta^{4r+2-k}\right)^n + \left(1+\alpha^{4r+2-k}\beta^k\right)^n \right]$$

$$= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left[\left(1+(-1)^k \beta^{4r+2-2k}\right)^n + \left(1+(-1)^k \alpha^{4r+2-2k}\right)^n \right]$$

$$= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{k(n+1)} \binom{4r+2}{k} \left[\left((-1)^k + \beta^{2(2r+1-k)}\right)^n + \left((-1)^k + \alpha^{2(2r+1-k)}\right)^n \right]. \tag{10}$$

We did not insert the middle term, since it is equal to

$$5^{-(2r+1)}(-1)^{2r+1}\binom{4r+2}{2r+1}(1+\alpha^{2r+1}\beta^{2r+1})^n$$

$$= 5^{-(2r+1)}(-1)^{2r+1}\binom{4r+2}{2r+1}(1+(-1)^{2r+1})^n = 0.$$

In (10), using (6), and observing that $\alpha^{2(2r+1-k)} + (-1)^k = \alpha^{2(2r+1-k)} - (-1)^{2r+1-k}$, we get

$$S_{4r+2,n}(1) = 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{(n+1)k} \binom{4r+2}{k} 5^{\frac{n}{2}} F^n_{2r+1-k} \left((-1)^n \beta^{n(2r+1-k)} + \alpha^{n(2r+1-k)} \right).$$

Therefore, if n is odd, then

$$S_{4r+2,n}(1) = 5^{-(2r+1)} \sum_{k=0}^{2r} \binom{4r+2}{k} 5^{\frac{n+1}{2}} F^n_{2r+1-k} F_{n(2r+1-k)}$$

and, if n is even, then

$$S_{4r+2,n}(1) = 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} 5^{\frac{n}{2}} F^n_{2r+1-k} L_{n(2r+1-k)}.$$

In the same way, associating $k \leftrightarrow 4r - k$, except for the middle term, and using Lemma 10, we get

$$S_{4r,n}(1) = 5^{-2r} \sum_{k=0}^{2r-1} (-1)^k \binom{4r}{k} \left[\left(1 + \alpha^k \beta^{4r-k}\right)^n + \left(1 + \alpha^{4r-k} \beta^k\right)^n \right] + 5^{-2r} 2^n \binom{4r}{2r}$$

$$= 5^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left[\left((-1)^k + \beta^{2(2r-k)}\right)^n + \left((-1)^k + \alpha^{2(2r-k)}\right)^n \right]$$

$$+ 5^{-2r} 2^n \binom{4r}{2r} \tag{11}$$

$$= 5^{-2r} \left[\sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left(L_{2r-k}^n \beta^{(2r-k)n} + L_{2r-k}^n \alpha^{(2r-k)n} \right) + 2^n \binom{4r}{2r} \right]$$

$$= 5^{-2r} \left[\sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} L_{2r-k}^n L_{(2r-k)n} + 2^n \binom{4r}{2r} \right]. \quad \square$$

Remark 12: *In the same manner we can find* $\sum_{i=0}^{n} \binom{n}{i} U_{pi}^r x^i$.

We now list some interesting special cases of Theorems 9 and 11.

Corollary 13: *We have*

$$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$$

$$\sum_{i=0}^{2n} \binom{2n}{i} F_i^2 = 5^{n-1} L_{2n}$$

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_i^2 = 5^n F_{2n+1}$$

$$\sum_{i=0}^{n} \binom{n}{i} F_i^3 = \frac{1}{5}(2^n F_{2n} + 3F_n)$$

$$\sum_{i=0}^{n} \binom{n}{i} F_i^4 = \frac{1}{25}(3^n L_{2n} - 4(-1)^n L_n + 6 \cdot 2^n).$$

Proof: The second, third and fifth identities follow from Theorem 11. Now, using Theorem 9, with $A = \frac{1}{\sqrt{5}}$, we get

$$S_{1,n}(1) = \frac{1}{\sqrt{5}}\sum_{k=0}^{1}(-1)^{1-k}\binom{1}{k}(1+\alpha^k\beta^{1-k})^n$$

$$= \frac{1}{\sqrt{5}}(-(1+\beta)^n + (1+\alpha)^n) = \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) = F_{2n}.$$

Next, the fourth identity follows from

$$S_{3,n}(1) = \frac{1}{5\sqrt{5}}\sum_{k=0}^{3}(-1)^{3-k}\binom{3}{k}(1+\alpha^k\beta^{3-k})^n$$

$$= \frac{1}{5\sqrt{5}}[-(1+\beta^3)^n + 3(1+\alpha\beta^2)^n - 3(1+\alpha^2\beta)^n + (1+\alpha^3)^n]$$

$$= \frac{1}{5\sqrt{5}}[-(2\beta^2)^n + 3\alpha^n - 3\beta^n + (2\alpha^2)^n] = \frac{1}{5}(2^nF_{2n} + 3F_n),$$

since $1+\beta^3 = 2\beta^2$, $1+\alpha^3 = 2\alpha^2$. $\square$

The results in our next theorem are obtained by putting $x=-1$ in Theorem 9, and since the proofs are similar to the proofs in Theorem 11, we omit them.

Theorem 14: *We have*

$$S_{4r,n}(-1) = 5^{\frac{n}{2}-2r}\sum_{k=0}^{2r-1}(-1)^k\binom{4r}{k}F^n_{2r-k}L_{(2r-k)n}, \textit{ if } n \textit{ is even},$$

$$S_{4r,n}(-1) = -5^{\frac{n+1}{2}-2r}\sum_{k=0}^{2r-1}\binom{4r}{k}F^n_{2r-k}F_{(2r-k)n}, \textit{ if } n \textit{ is odd},$$

$$S_{4r+2,n}(-1) = 5^{-(2r+1)}\left[\sum_{k=0}^{2r}(-1)^{k(n+1)+n}\binom{4r+2}{k}L^n_{2r+1-k}L_{(2r+1-k)n} - 2^n\binom{4r+2}{2r+1}\right].$$

Next we record some interesting special cases of Theorem 9 and 14.

Corollary 15: *We have*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i=-F_n$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^2=\frac{1}{5}((-1)^nL_n-2^{n+1})$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^3=\frac{1}{5}((-2)^nF_n-3F_{2n})$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^4=5^{\frac{n-4}{2}}(L_{2n}-4L_n),\ \textit{if } n \textit{ is even}$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^4=-5^{\frac{n-3}{2}}(F_{2n}+4F_n),\ \textit{if } n \textit{ is odd.}$$

Proof: The first identity is a simple application of Theorem 9. The identities for even powers are immediate consequences of Theorem 14. Now, using Theorem 9, we get

$$\begin{aligned}S_{3,n}(-1)&=\frac{1}{5\sqrt{5}}(-(1-\beta^3)^n+3(1-\alpha\beta^2)^n-3(1-\alpha^2\beta)^n+(1-\alpha^3)^n)\\&=\frac{1}{5\sqrt{5}}(-(-2)^n\beta^n+3\beta^{2n}-3\alpha^{2n}+(-2)^n\alpha^n)=\frac{1}{5}((-2)^nF_n-3F_{2n}),\end{aligned}$$

since $1-\beta^3=-2\beta,\ 1-\alpha^3=-2\alpha$. □

From (9) we obtain, for $r\geq 1$,

$$\sum_{k=0}^{2r-1}(-1)^{k(n+1)}\binom{4r}{k}L_{2r-k}^nL_{(2r-k)n}+2^n\binom{4r}{2r}\equiv 0 \pmod{5^{2r}}.$$

Similar congruence results follow from other sums in Section 4, and we leave these for the reader to formulate.

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