ON THE k^{th}-ORDER F-L IDENTITY

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1. INTRODUCTION

For convenience, in this paper we adopt the notations and symbols in [3] or [4]:
Let the sequence $\{w_n\}$ be defined by the recurrence relation

$$w_{n+k} = a_1 w_{n+k-1} + \cdots + a_{k-1} w_{n+1} + a_k w_n, \tag{1.1}$$

and the initial conditions

$$w_0 = c_0, w_1 = c_1, \ldots, w_{k-1} = c_{k-1}, \tag{1.2}$$

where $a_1, \ldots, a_k$, and $c_0, \ldots, c_{k-1}$ are complex constants. Then we call $\{w_n\}$ a k^{th}- **order Fibonacci-Lucas sequence** or simply an **F-L sequence**, call every w_n an **F-L number**, and call

$$f(x) = x^k - a_1 x^{k-1} - \cdots - a_{k-1} x - a_k \tag{1.3}$$

the **characteristic polynomial** of $\{w_n\}$. A number α satisfying $f(\alpha) = 0$ is called a **characteristic root** of $\{w_n\}$. In this paper we always assume that $a_k \neq 0$, hence we may consider $\{w_n\}$ as $\{w_n\}_{-\infty}^{+\infty}$. The set of F-L sequences satisfying (1.1) is denoted by $\Omega(a_1, \ldots, a_k)$ and also by $\Omega(f(x))$. Let $x_1, \ldots, x_k$ be the roots of $f(x)$ defined by (1.3), and let

$$v_n = x_1^n + x_2^n + \cdots + x_k^n (n \in \mathbb{Z}). \tag{1.4}$$

Then, obviously, $\{v_n\} \in \Omega(a_1, \ldots, a_k)$. Since for $k = 2$ and $a_1 = a_2 = 1, \{v_n\}$ is just the classical Lucas sequence $\{L_n\}$, we call $\{v_n\}$ for any k the k^{th}**-order Lucas sequence** in $\Omega(a_1, \ldots, a_k)$. In [1] and [2] Howard proved the following theorem:

Theorem 1.1: *Let $\{w_n\} \in \Omega(a_1, \ldots, a_k)$. Then for $m \geq 1$ and all integers n,*

$$w_{(k-1)m+n} = \sum_{j=1}^{k} (-1)^{j-1} c_{m,jm} w_{(k-j-1)m+n}.$$

The numbers $c_{m,jm}$ are defined by

$$\prod_{i=0}^{m-1} [1 - a_1(\theta^i x) - a_2(\theta^i x)^2 - \cdots - a_k(\theta^i x)^k] = 1 + \sum_{j=1}^{k} (-1)^j c_{m,jm} x^{jm},$$

where θ is a primitive m^{th} root of unity.

Yet in [2] he proved the following result:

Theorem 1.2: *Let $\{w_n\} \in \Omega(r,s,t)$. Then for $m, n \in \mathbb{Z}$,*

$$w_{n+2m} = J_m w_{n+m} - t^m J_{-m} w_n + t^m w_{n-m}. \tag{1.5}$$

Here $\{J_n\} \in \Omega(r,s,t)$ satisfies $J_0 = 3, J_1 = r, J_2 = r^2 + 2s$.

It is easy to see that $\{J_n\}$ is just the third-order Lucas sequence in $\Omega(r,s,t)$. Thus we observe that the identity (1.5) involves only the numbers from an arbitrary third-order F-L sequence and from the third-order Lucas sequence in $\Omega(r,s,t)$. This suggests the main purpose of the present paper: we shall prove a general k^{th}-order F-L identity which involves only the numbers from an arbitrary k^{th}-order F-L sequence and from the k^{th}-order Lucas sequence in $\Omega(a_1, \ldots, a_k)$. As an application of the identity we represent $c_{m,jm}$ in Theorem 1.1 by the k^{th}-order Lucas numbers. Then to make the identity simpler we give the identity an alternative form in which the negative subscripts for the k^{th}-order Lucas sequence are introduced. As a corollary of the identity we generalize the result of Theorem 1.2 from the case $k = 3$ to the case of any k. In our proofs we do not need to consider whether the characteristic roots of the F-L sequence are distinct. Also, we can use our results to construct identities for given k, and the computations are relatively simple. We first give some preliminaries in Section 2, and then in Section 3 we give the main results and their proofs. Some examples are also given in Section 3.

2. PRELIMINARIES

Lemma 2.1: *Let $\{v_n\}$ be the k^{th}-order Lucas sequence in $\Omega(a_1, \ldots, a_k)$. Denote the generating function of $\{v_n\}$ by*

$$V(x) = \sum_{n=0}^{\infty} v_n x^n. \tag{2.1}$$

Then

$$V(x) = \frac{k - (k-1)a_1 x - (k-2)a_2 x^2 - \cdots - 2a_{k-2}x^{k-2} - a_{k-1}x^{k-1}}{1 - a_1 x - a_2 x^2 - \cdots - a_k x^k}. \tag{2.2}$$

Proof: Let $x_1, \ldots, x_k$ be the roots of the characteristic polynomial $f(x)$, denoted by (1.3), of sequence $\{v_n\}$. Denote

$$f^*(x) = 1 - a_1 x - a_2 x^2 - \cdots - a_k x^k.$$

Clearly,

$$f^*(x) = x^k f(x^{-1}) = (1 - x_1 x) \ldots (1 - x_k x).$$

Whence

$$\ln f^*(x) = \ln(1 - x_1 x) + \cdots + \ln(1 - x_k x).$$

Differentiating the both sides of the last expression we obtain

$$\frac{f^{*'}(x)}{f^*(x)} = \frac{-x_1}{1-x_1x} + \cdots + \frac{-x_k}{1-x_kx}$$

$$= -\sum_{n=0}^{\infty}(x_1^{n+1} + \cdots + x_k^{n+1})x^n = -\sum_{n=0}^{\infty} v_{n+1}x^n.$$

From (2.1) it follows that

$$V(x) = v_0 - x \cdot \frac{f^{*'}(x)}{f^*(x)} = k + \frac{x(a_1 + 2a_2x + \cdots + ka_kx^{k-1})}{1 - a_1x - a_2x^2 - \cdots - a_kx^k}.$$

Thus the proof is finished. □

From (2.1) and (2.2) it follows that

$$(1 - a_1x - a_2x^2 - \cdots - a_kx^k)\sum_{n=0}^{\infty} v_nx^n$$

$$= k - (k-1)a_1x - (k-2)a_2x^2 - \cdots - 2a_{k-2}x^{k-2} - a_{k-1}x^{k-1}.$$

Comparing the coefficients of x^i in the both sides of the last expression for $i = 1, \ldots, k$ we get the well-known Newton's formula:

Corollary 2.2: **(Newton's formula)** *Let $\{v_n\}$ be the k^{th}-order Lucas sequence in $\Omega(a_1, \ldots, a_k)$. Then*

$$a_1v_{i-1} + a_2v_{i-2} + \cdots + a_{i-1}v_1 + ia_i = v_i \quad (i = 1, \ldots, k).$$

Lemma 2.3: [4] *Let $\{w_n\} \in \Omega(a_1, \ldots, a_k) = \Omega(f(x))$, and $x_1, \ldots, x_k$ be the roots of $f(x)$. For $m \in \mathbb{Z}^+$, let*

$$f_m(x) = (x - x_1^m)\ldots(x - x_k^m) = x^k - b_1x^{k-1} - \cdots - b_{k-1}x - b_k. \tag{2.3}$$

Then $\{w_{mn+r}\}_n \in \Omega(f_m(x))$. That is,

$$w_{m(n+k)+r} = b_1w_{m(n+k-1)+r} + \cdots + b_{k-1}w_{m(n+1)+r} + b_kw_{mn+r}.$$

3. THE MAIN RESULTS AND PROOFS

Theorem 3.1: *Let $\{w_n\}$ be any sequence in $\Omega(a_1, \dots, a_k) = \Omega(f(x))$, and let $\{v_n\}$ be the k^{th}-order Lucas sequence in $\Omega(f(x))$. Let $x_1, \dots, x_k$ be the roots of $f(x)$ and $f_m(x)$ be defined by (2.3) for $m \in \mathbb{Z}^+$. Then for $n \in \mathbb{Z}$,*

$$w_{m(n+k)+r} = b_1 w_{m(n+k-1)+r} + \cdots + b_{k-1} w_{m(n+1)+r} + b_k w_{mn+r}, \tag{3.1}$$

and $b_1, \dots, b_k$ can be obtained by solving the trianglular system of linear equations

$$b_1 v_{m(i-1)} + b_2 v_{m(i-2)} + \cdots + b_{i-1} v_m + i b_i = v_{mi} \quad (i = 1, \dots, k). \tag{3.2}$$

In other words, for $i = 1, \dots, k$,

$$b_i = b_i(m) = \frac{1}{i!} \begin{vmatrix} 1 & & & & & & v_m \\ v_m & 2 & & & & & v_{2m} \\ v_{2m} & v_m & 3 & & & & v_{3m} \\ v_{3m} & v_{2m} & v_m & \ddots & & & v_{4m} \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \vdots \\ v_{(i-2)m} & v_{(i-3)m} & v_{(i-4)m} & \cdots & v_m & i-1 & v_{(i-1)m} \\ v_{(i-1)m} & v_{(i-2)m} & v_{(i-3)m} & \cdots & v_{2m} & v_m & v_{im} \end{vmatrix}. \tag{3.3}$$

Proof: In $\Omega(f_m(x))$ the k^{th}-order Lucas sequence is

$$V_n = (x_1^m)^n + \cdots + (x_k^m)^n = v_{mn} (n \in \mathbb{Z}).$$

Thus (3.1) and (3.2) follow from Lemma 2.3 and Corollary 2.2. We use Cramer's Rule on (3.2) to obtain (3.3) □

Remark: In (3.1) taking $n = -1$ and then taking $r = n$ we get $c_{m,jm} = b_j(m)$. Then $c_{m,jm}$ can be represented by the k^{th}-order Lucas numbers and it is more easy to caluclate $c_{m,jm}$'s. For example, by using (3.2) or (3.3) we can obtain:

For $k = 3$,

$$w_{n+2m} = v_m w_{n+m} + (v_{2m} - v_m^2)/2 \cdot w_n + (2v_{3m} - 3v_m v_{2m} + v_m^3)/6 \cdot w_{n-m};$$

For $k = 4$,

$$\begin{aligned} w_{n+3m} =& v_m w_{n+2m} + (v_{2m} - v_m^2)/2 \cdot w_{n+m} + (2v_{3m} - 3v_m v_{2m} + v_m^3)/6 \cdot w_n + \\ & (6v_{4m} - 8v_m v_{3m} - 3v_{2m}^2 + 6v_m^2 v_{2m} - v_m^4)/24 \cdot w_{n-m}; \end{aligned}$$

For $k = 5$,

$$\begin{aligned} w_{n+4m} =& v_m w_{n+3m} + (v_{2m} - v_m^2)/2 \cdot w_{n+2m} + (2v_{3m} - 3v_m v_{2m} + v_m^3)/6 \cdot w_{n+m} + \\ & (6v_{4m} - 8v_m v_{3m} - 3v_{2m}^2 + 6v_m^2 v_{2m} - v_m^4)/24 \cdot w_n + \\ & (24v_{5m} - 30v_m v_{4m} - 20v_{2m} v_{3m} + 20v_m^2 v_{3m} + 15v_m v_{2m}^2 - \\ & 10v_m^3 v_{2m} + v_m^5)/120 \cdot w_{n-m}. \end{aligned}$$

Theorem 3.2: *Under the conditions of Theorem 3.1 we have*

$$b_{k-i}(m) = (-1)^{(k+1)(m+1)+1} a_k^m b_i(-m) \quad (i = 1, \ldots, k-1), \tag{3.4}$$

and

$$b_k(m) = (-1)^{(k+1)(m+1)} a_k^m. \tag{3.5}$$

Therefore for odd k we have

$$\begin{aligned} w_{m(n+k)+r} =& b_1(m)w_{m(n+k-1)+r} + b_2(m)w_{m(n+k-2)+r} + \cdots + \\ & b_{(k-1)/2}(m)w_{m(n+(k+1)/2)+r} - a_k^m(b_{(k-1)/2}(-m)w_{m(n+(k-1)/2)+r} + \\ & \cdots + b_2(-m)w_{m(n+2)+r} + b_1(-m)w_{m(n+1)+r} - w_{mn+r}), \end{aligned} \tag{3.6}$$

and for even k we have

$$\begin{aligned} w_{m(n+k)+r} =& b_1(m)w_{m(n+k-1)+r} + b_2(m)w_{m(n+k-2)+r} + \cdots + \\ & b_{k/2-1}(m)w_{m(n+k/2+1)+r} + b_{k/2}(m)w_{m(n+k/2)+r} + \\ & (-a_k)^m(b_{k/2-1}(-m)w_{m(n+k/2-1)+r} + \\ & \cdots + b_2(-m)w_{m(n+2)+r} + b_1(-m)w_{m(n+1)+r} - w_{mn+r}). \end{aligned} \tag{3.7}$$

Proof: Clearly,

$$b_k = b_k(m) = -(-1)^k x_1^m \ldots x_k^m = (-1)^{k+1}(-(-1)^k a_k)^m.$$

Whence (3.5) holds. Let

$$\begin{aligned} f_m^*(x) = x^k f_m(x^{-1}) &= (1 - x_1^m x) \ldots (1 - x_k^m x) \\ &= 1 - b_1 x - b_2 x^2 - \cdots - b_{k-1}x^{k-1} - b_k x^k. \end{aligned}$$

Then the k^{th}-order Lucas sequence in $\Omega(f_m^*(x))$ is

$$V_n^* = (x_1^{-m})^n + \cdots + (x_k^{-m})^n = v_{-mn} (n \in \mathbb{Z}).$$

By Newton's formula we have, for $i = 1, \ldots, k-1$,

$$b_{k-1}v_{-m(i-1)} + b_{k-2}v_{-m(i-2)} + \cdots + b_{k-(i-1)}v_{-m} + ib_{k-i} = -b_k v_{-mi},$$

where $b_i = b_i(m)$. It follows from Cramer's Rule that

$$b_{k-i} = b_{k-i}(m) = \frac{-b_k(m)}{i!} \begin{vmatrix} 1 & & & & & & v_{-m} \\ v_{-m} & 2 & & & & & v_{-2m} \\ v_{-2m} & v_{-m} & 3 & & & & v_{-3m} \\ v_{-3m} & v_{-2m} & v_{-m} & \ddots & & & v_{-4m} \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \vdots \\ v_{-(i-2)m} & v_{-(i-3)m} & v_{-(i-4)m} & \cdots & v_{-m} & i-1 & v_{-(i-1)m} \\ v_{-(i-1)m} & v_{-(i-2)m} & v_{-(i-3)m} & \cdots & v_{-2m} & v_{-m} & v_{-im} \end{vmatrix}. \tag{3.8}$$

Noticing (3.5) and comparing (3.8) with (3.3) we see that (3.4) holds. This completes the proof. □

Corollary 3.3: *Let $\{w_n\}$ be any sequence in $\Omega(a_1, \ldots, a_k\} = \Omega(f(x))$, and let $\{v_n\}$ be the k^{th}-order Lucas sequence in $\Omega(f(x))$. Assume that $n, m \in Z$ and $m \neq 0$. Then, for odd k we have*

$$\begin{aligned} w_{n+m(k-1)} =& b_1(m)w_{n+m(k-2)} + b_2(m)w_{n+m(k-3)} + \cdots + \\ & b_{(k-1)/2}(m)w_{n+m(k-1)/2} - a_k^m(b_{(k-1)/2}(-m)w_{n+m(k-3)/2} + \\ & \cdots + b_2(-m)w_{n+m} + b_1(-m)w_n - w_{n-m}), \end{aligned} \tag{3.9}$$

and for even k we have

$$\begin{aligned} w_{n+m(k-1)} =& b_1(m)w_{n+m(k-2)} + b_2(m)w_{n+m(k-3)} + \cdots + \\ & b_{k/2-1}(m)w_{n+mk/2} + b_{k/2}(m)w_{n+m(k/2-1)} + \\ & (-a_k)^m(b_{k/2-1}(-m)w_{n+m(k/2-2)} + \\ & \cdots + b_2(-m)w_{n+m} + b_1(-m)w_n - w_{n-m}). \end{aligned} \tag{3.10}$$

Proof: For $m > 0$ the conclusion is shown by taking $n = -1$ and then taking $r = n$ in Theorem 3.2. Now, assume that $m < 0$. Then, $-m > 0$, and, by the proved result, for odd k we have

$$\begin{aligned} w_{n-m(k-1)} =& b_1(-m)w_{n-m(k-2)} + b_2(-m)w_{n-m(k-3)} + \cdots + \\ & b_{(k-1)/2}(-m)w_{n-m(k-1)/2} - a_k^{-m}(b_{(k-1)/2}(m)w_{n-m(k-3)/2} + \\ & \cdots + b_2(m)w_{n-m} + b_1(m)w_n - w_{n+m}), \end{aligned} \tag{3.11}$$

and for even k we have

$$\begin{aligned} w_{n-m(k-1)} =& b_1(-m)w_{n-m(k-2)} + b_2(-m)w_{n-m(k-3)} + \cdots + \\ & b_{k/2-1}(-m)w_{n-mk/2} + b_{k/2}(-m)w_{n-m(k/2-1)} + \\ & (-a_k)^{-m}(b_{k/2-1}(m)w_{n-m(k/2-2)} + \\ & \cdots + b_2(m)w_{n-m} + b_1(m)w_n - w_{n+m}). \end{aligned} \tag{3.12}$$

Multiplying both sides of (3.11) by a_k^m and replacing n by $n + m(k-2)$ we can get (3.9). Multiplying both sides of (3.12) by $(-a_k)^m$ and replacing n by $n+m(k-2)$ we can get (3.10). Thus the proof is finished. □

Remark: Corollary 3.3 is a generalization of Theorem 1.2 (for $k = 3$). By using the corollary we can easily give the following examples:

For $k = 4$,

$$w_{n+3m} = v_m w_{n+2m} + (v_{2m} - v_m^2)/2 \cdot w_{n+m} + (-a_4)^m (v_{-m} w_n - w_{n-m});$$

For $k = 5$,

$$\begin{aligned} w_{n+4m} =& v_m w_{n+3m} + (v_{2m} - v_m^2)/2 \cdot w_{n+2m} - \\ & a_5^m ((v_{-2m} - v_{-m}^2)/2 \cdot w_{n+m} + v_{-m} w_n - w_{n-m}); \end{aligned}$$

For $k = 6$,

$$\begin{aligned} w_{n+5m} =& v_m w_{n+4m} + (v_{2m} - v_m^2)/2 \cdot w_{n+3m} + (2v_{3m} - 3v_m v_{2m} + v_m^3)/6 \cdot w_{n+2m} + \\ & (-a_6)^m ((v_{-2m} - v_{-m}^2)/2 \cdot w_{n+m} + v_{-m} w_n - w_{n-m}). \end{aligned}$$

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