ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-603 Proposed by the E. Herrmann, Siegburg, Germany

Show that if $n \geq 3$ and $n \equiv 1 \pmod 2$, then

$$\frac{1}{F_n} < \sum_{k=0}^{\infty} \frac{1}{F_{n+2k}} < \frac{1}{F_{n-1}}.$$

However, if $n \geq 4$ and $n \equiv 0 \pmod 2$, then

$$\frac{1}{F_{n-1}} < \sum_{k=0}^{\infty} \frac{1}{F_{n+2k}} < \frac{1}{F_{n-2}}.$$

H-604 Proposed by Mario Catalani, Torino, Italy

In **H-592**, the proposers introduced, for $n \geq 2$, a nondiagonal $n \times n$ matrix A such that $A^2 = xA + yI$, where x, y are indeterminates and I is the identity matrix.

a) State the conditions under which all the eigenvalues of A are equal.

b) Assume now that not all the eigenvalues of A are equal. Assume that A is a $2n \times 2n$ matrix, and that $\operatorname{tr}(A) = nx$. Consider the Hamilton-Cayley equation for A

$$\sum_{k=0}^{2n} (-1)^k \lambda_k A^{2n-k} = 0,$$

where $\lambda_0 = 1$. Find $\sum_{k=0}^{2n} \lambda_k$.

H-605 Proposed by José Luis Díaz-Barrero & Juan José Egozcue, Barcelona, Spain

Find the smallest integer k for which $\lambda_0 a_n + \lambda_1 a_{n-1} + \cdots + \lambda_k a_{n+k} = 0$ holds for all $n \geq 1$ with some integers $\lambda_0, \ldots, \lambda_k$ not all zero, where $\{a_n\}_{n \geq 1}$is the integer sequence defined by

$$a_n = \Big(\sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2\ell+1} 2^{\ell} \Big) \Big(\sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2^{n-1}} \binom{n}{2\ell+1} 5^{\ell} \Big).$$

SOLUTIONS

Some properties of the number 5

H-591 Proposed by H.-J. Sieffert, Berlin, Germany
(Vol. 40, no. 5, November 2002)

Prove that, for all positive integers n,

(a)

$$5^n F_{2n-1} = \sum_{\substack{k=0 \\ 5 \not| 2n-k+3}}^{2n} (-1)^{\lfloor (4n+3k)/5 \rfloor} \binom{4n+1}{k},$$

(b)

$$5^n L_{2n} = \sum_{\substack{k=0 \\ 5 \not| 2n-k+4}}^{2n+1} (-1)^{\lfloor (4n+3k-3)/5 \rfloor} \binom{4n+3}{k},$$

(c)

$$5^{n-1} F_{2n} = \sum_{\substack{k=0 \\ 5 \not| 2n-k+1}}^{2n-2} (-1)^{\lfloor (8n+k+3)/5 \rfloor} \binom{4n-3}{k},$$

(d)

$$5^{n-1} L_{2n+1} = \sum_{\substack{k=0 \\ 5 \not| 2n-k+2}}^{2n-1} \binom{4n-1}{k},$$

where $\lfloor \ \rfloor$ denotes the greatest integer function.

Solution by the proposer

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{k+2}(x) = xF_{k+1}(x) + F_k(x)$ for $k \geq 0$. From **H-492**, we know that, for all complex numbers x and y and all nonegative integers n,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} F_{n-2k}(x) F_{n-2k}(y) = z^{n-1} F_n(xy/z),$$

where $z = \sqrt{x^2 + y^2 + 4}$. Replacing n by $2n+1$ and taking $y = 0$, after a suitable reindexing, we obtain

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} F_{2k+1}(x) = (x^2+4)^n. \tag{1}$$

Let $B_k = (-1)^k F_{2k+1}(\,\mathrm{i}\alpha)$, $k \geq 0$, where $\mathrm{i} = \sqrt{-1}$ and α is the golden section. Then, the sequence $\{B_k\}_{k\geq 0}$ satisfies the recursion $B_{k+2} = -\beta B_{k+1} - B_k$ for $k \geq 0$, where β is the conjugate of α, and a simple induction argument shows that

$$B_k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 5, \\ \alpha & \text{if } k \equiv 1 \pmod 5, \\ 0 & \text{if } k \equiv 0 \pmod 5, \\ -\alpha & \text{if } k \equiv 0 \pmod 5, \\ -1 & \text{if } k \equiv 0 \pmod 5. \end{cases} \tag{2}$$

Since $4-\alpha^2=-\sqrt{5}\beta$, identity (1) with $x=$ iα gives

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}B_k=(-1)^n5^{n/2}\beta^n. \tag{3}$$

Define the sequences $\{c_k\}_{k\geq 0}$ and $\{d_k\}_{k\geq 0}$ by

$$c_k=\begin{cases}1 & if k\equiv 0 \pmod 5,\\ -1 & if k\equiv 4 \pmod 5,\\ 0 & \text{otherwise},\end{cases}\qquad \text{and}\qquad d_k=\begin{cases}1 & if k\equiv 1 \pmod 5,\\ -1 & if k\equiv 3 \pmod 5,\\ 0 & \text{otherwise},\end{cases}$$

and let

$$S_n=\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}c_k \qquad \text{and} \qquad T_n=\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}d_k.$$

Then, by (2) and (3), $S_n+\alpha T_n=(-1)^n5^{n/2}\beta^n$. Since $2\beta^n=L_n-\sqrt{5}F_n$, we then have

$$2S_n+2\alpha T_n=\begin{cases}5^{n/2}L_n-5^{(n+1)/2}F_n & \text{if } n \text{ is even},\\ \\ 5^{(n+1)/2}F_n-5^{n/2}L_n & \text{if } n \text{ is odd}.\end{cases}$$

Using $2\alpha=1+\sqrt{5}$ together with the fact $\sqrt{5}$ is irrational, we then must have

$$2S_n+T_n=\begin{cases}5^{n/2}L_n & \text{if } n \text{ is even},\\ \\ 5^{(n+1)/2}F_n & \text{if } n \text{ is odd},\end{cases} \tag{4}$$

and

$$T_n=\begin{cases}-5^{n/2}F_n & \text{if } n \text{ is even},\\ \\ -5^{(n-1)/2}L_n & \text{if } n \text{ is odd}.\end{cases} \tag{5}$$

Substracting (5) from (4) yields

$$2S_n=\begin{cases}5^{n/2}(F_n+L_n) & \text{if } n \text{ is even},\\ \\ 5^{(n-1)/2}(5F_n+L_n) & \text{if } n \text{ is odd}.\end{cases} \tag{6}$$

Dividing (6) by 2, subtracting the resulting equation from (4), and noting that $L_n-F_n=2F_{n-1}$ and $5F_n-L_n=2L_{n-1}$, we find

$$S_n+T_n=\begin{cases}5^{n/2}F_{n-1} & \text{if } n \text{ is even},\\ \\ 5^{(n-1)/2}L_{n-1} & \text{if } n \text{ is odd}.\end{cases} \tag{7}$$

On the other hand, from

$$c_k + d_k = \begin{cases} (-1)^{\lfloor 2k/5 \rfloor} & \text{if } k \not\equiv 2 \pmod 5, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$S_n + T_n = \sum_{\substack{k=0 \\ 5 \nmid k+3}}^{n} (-1)^{\lfloor 7k/5 \rfloor} \binom{2n+1}{n-k}. \tag{8}$$

The desired identities (a) and (b) now easily follow from (7) and (8) by repacing n by $2n$ (respectively, by $2n+1$), and reindexing.
Multiplying (5) by 3, substracting the resulting equation from (4), and noting that $L_n + 3F_n = 2F_{n+2}$ and $5F_n + 3L_n = 2L_{n+2}$, we obtain

$$S_n - T_n = \begin{cases} 5^{n/2} F_{n+2} & \text{if } n \text{ is even,} \\ \\ 5^{(n-1)/2} L_{n+2} & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

Since

$$c_k - d_k = \begin{cases} (-1)^{\lfloor (4k+1)/5 \rfloor} & \text{if } k \not\equiv 2 \pmod 5, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$S_n - T_n = \sum_{\substack{k=0 \\ 5 \nmid k+3}}^{n} (-1)^{\lfloor (9k+1)/5 \rfloor} \binom{2n+1}{n-k}. \tag{10}$$

The desired identities (c) and (d) now easily follow from (9) and (10) by repacing n by $2n-2$ (respectively, by $2n-1$), and reindexing.

Also solved by Paul Bruckman and Vincent Mathe.

Matrices satisfying quadratic equations

H-592 Proposed by N. Gautheir & J.B. Gosselin, Royal Military College of Canada (Vol. 40, no. 5, November 2002)

For integers $m \geq 1$, $n \geq 2$, let X be a nontrivial $n \times n$ matrix such that

$$X^2 = xX + yI, \tag{1}$$

where x, y are indeterminates and I is a unit matrix. (By definition, a trivial matrix is diagonal.) Then consider the Fibonacci and Lucas sequences of polynomials, $\{F_l(x,y)\}_{l=0}^{\infty}$ and $\{L_l(x,y)\}_{l=0}^{\infty}$, defined by the recurences

$$F_0(x,y) = 0,\ F_1(x,y) = 1,\ F_{l+2}(x,y) = xF_{l+1}(x,y) + yF_l(x,y), \tag{2}$$

$$L_0(x,y) = 2,\ L_1(x,y) = x,\ L_{l+2}(x,y) = xL_{l+1}(x,y) + yL_l(x,y), \tag{3}$$

respectively.

a. Show that

$$X^m = a_m X + b_m yI \qquad \text{and that} \qquad X^m + (-y)^m X^{-m} = c_m I,$$

where a_m, b_m, and c_m are to be expressed in closed form as functions of the polynomials (2).

b. Now let

$$f(\lambda; x, y) \equiv |\lambda I - X| \equiv \sum_{m=0}^{n} (-1)^{n-m} \lambda_{n-m} \lambda^m$$

be the characteristic (monic) polynomial associated to X, where the set of coefficients,

$$\{\lambda_l \equiv \lambda_l(x,y); 0 \le l \le n\}$$

is entirely determined from the defining relation for $f(\lambda; x, y)$. For example, $\lambda_0 = 1$, $\lambda_1 = \operatorname{tr}(X)$, $\lambda_n = \det(X)$, etc. Show that

$$\sum_{m=1}^{n} (-1)^m \lambda_{n-m} F_m(x,y) = 0 \qquad \text{and that} \qquad y \sum_{m=1}^{n} (-1)^m \lambda_{n-m} F_{m-1}(x,y) + \lambda_n = 0.$$

Solution by the proposers

a. First note that X has an inverse since $X(X - xX) = yI$ implies $\det(X) \neq 0$ (here, y is assumed to be a nonzero indeterminate). We prove by induction on $m \ge 1$ that

$$X^m = F_m(x,y)X + yF_{m-1}(x,y)I, \tag{3}$$

and that

$$X^m + (-y)^m X^{-m} = L_m(x,y)I, \tag{4}$$

so $a_m = F_m(x,y)$, $b_m = F_{m-1}(x,y)$ and $c_m = L_m(x,y)$. It is clear that (3) is true for $m = 1$ and 2. Now assume its validity for an arbitrary value of m and multiply (3) by X to get

$$X^{m+1} = F_m(x,y)X^2 + yF_{m-1}X = F_m(x,y)(xX + yI) + yF_{m-1}(x,y)X$$

$$= (xF_m(x,y) + yF_{m-1}(x,y))X + yF_m(x,y)I = F_{m+1}(x,y)X + yF_m(x,y)I,$$

which is formula (3) for $m+1$. To prove (4), note that it is true for $m = 1$ since (1) implies $X - yX^{-1} = xI$. Squaring this last result then shows that (4) also holds for $m = 2$. Now assume that (4) holds for $m \ge 2$ and multiply it by $X - yX^{-1} = xI$ to get

$$(X^{m+1} + (-y)^{m+1} X^{-(m+1)}) + (-yX^{m-1} + (-y)^m X^{-(m-1)}) = xL_m(x,y)I,$$

i.e.,

$$X^{m+1} + (-y)^{m+1} X^{-(m+1)} = y(X^{m-1} + (-y)^{m-1} X^{-(m-1)}) + xL_m(x,y)I$$

$$= yL_{m-1}(x,y)I + xL_m(x,y)I = L_{m+1}(x,y)I,$$

which proves that (4) holds for $m+1$ as well.

b. According to the Hamilton-Cayley theorem, if $f(\lambda; x, y)$ is the characteristic polynomial associated with the matrix X, then $f(X; x, y) = 0$. Consequently, upon cancelling out an overall factor of $(-1)^n$ and upon using (3) for X^m, we find that

$$0 = \sum_{m=0}^{n} (-1)^m \lambda_{n-m} X^m = \sum_{m=1}^{n} (-1)^m \lambda_{n-m} (F_m(x,y)X + yF_{m-1}(x,y)I) + \lambda_n I,$$

which leads to the formulae given in the statement of the problem when X and I are linearly independent, i.e., in nontrivial cases.

Vincent Mathe points out that that in the case $y = 0$ the matrix X is not necessarily invertible; see for example, the Solution of **H-578**, vol. 40, pages 474–476.

Also solved by Paul Bruckman, Mario Catalani, Toufik Mansour and Vincent Mathe.

A Lucas prime congruence

H-593 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 41, no. 1, February 2003)

Let $p > 5$ be a prime. Prove the congruence

$$2\sum_{k=0}^{\lfloor (p-5)/10\rfloor} \frac{(-1)^k}{2k+1} \equiv (-1)^{(p-1)/2}\frac{2^{p-1}-L_p}{p} \pmod{p}.$$

Solution by the proposer

It is wellknown that $L_p \equiv 1 \pmod{p}$. Since by Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$, we see that the expression appearing on the right hand side of the desired congruence is an integer.

From **H-562**, we know that, for all nonnegative integers n,

$$5\sum_{k=0}^{\lfloor (n-2)/5\rfloor} \binom{2n+1}{n-5k-2} = 4^n - L_{2n+1}. \tag{1}$$

If k is an integer such that $0 \le k \le \lfloor (p-5)/10 \rfloor$, then

$$p > \frac{p-1}{2} - 5k - 2 \ge \frac{p-1}{2} - 5\left\lfloor \frac{p-5}{10} \right\rfloor - 2 > \frac{p-1}{2} - 5\cdot\frac{p-5}{10} - 2 = 0,$$

because $(p-5)/10$ is not an integer. Since, as is known,

$$\frac{1}{p}\binom{p}{j} \equiv \frac{(-1)^{j-1}}{j} \pmod{p} \qquad \text{for } j = 1, \dots, p-1,$$

relation (1) with $n = (p-1)/2$ gives

$$5\sum_{k=0}^{\lfloor (p-5)/10\rfloor} \frac{(-1)^{(p+1)/2+k}}{(p-1)/2-5k-2} \equiv \frac{2^{p-1}-L_p}{p} \pmod{p}.$$

Multiplying by $(-1)^{(p-1)/2}$ and noting that

$$\frac{-5}{(p-1)/2-5k-2} = \frac{-10}{p-10k-5} \equiv \frac{2}{2k+1} \pmod{p}$$

gives the desired congruence.

Also solved by Paul Bruckman.

Please Send in Proposals!