INTERVAL-FILLING SEQUENCES INVOLVING RECIPROCAL FIBONACCI NUMBERS

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1. INTRODUCTION

Let $r > 0$ be a fixed real number. In this paper we will study infinite series of the form:

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{(F_n)^r} \; (\epsilon_n = 1 \text{ or } 0), \tag{1}$$

where $x \in [0, I_r]$. I_r signifies the sum of series (1) if $\epsilon_n = 1$ for all $n \in N$. The convergence of the series (1), if $x = I_r$, can be easily proved by the well-known Binet formula! Letting $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ we have

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}. \tag{2}$$

Notice that $0 < \alpha^{-r} < 1$ and that Binet's formula yields $\lim_{n\to\infty}(F_n)^r/\alpha^{rn} = (\sqrt{5})^{-r}$. Thus applying the quotient-criterion for infinite series and geometric series proves the convergence of (1). For example: $I_1 = 3,359\ldots$. Furthermore it is easy to see that

$$I_r > I_{r'} \text{ for } r < r', \; I_r \to \infty \text{ for } r \to 0 \text{ and } I_r \to 2 \text{ for } r \to \infty. \tag{3}$$

We begin with certain results due to J.L. Brown in [1] and P. Ribenboim in [8] dealing with the representation of real numbers in the form (1). In [1] J.L. Brown treated the case $r = 1$. In [8] P. Ribenboim proved that for every positive real number x there exists a unique integer $m \geq 1$ such that $I_{1/(m-1)} < x \leq I_{1/m}$ and x is representable in the form (1) with $r = 1/m$, but x is not of the form (1) with $r = 1/(m-1)$ because $x > I_{1/(m-1)} (I_\infty = 0)$. Besides requiring $r > 0$ we do not make any other restrictions on r.

The following theorem is basic for our considerations.

Theorem 1: (S. Kakeya, 1914) Let (λ_n) be a sequence of positive real numbers, such that the series

$$\sum_{n=1}^{\infty} \lambda_n = s \tag{4}$$

is convergent with sum s and the inequalities

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \tag{5}$$

are fulfilled.

Then, each number $x \in [0, s]$ may be written in the form

$$x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n \qquad \epsilon_n \in \{0, 1\} \tag{6}$$

if and only if

$$\lambda_n \leq \lambda_{n+1} + \lambda_{n+2} + \dots \tag{7}$$

for all $n \in N$.

The "digits" ϵ_n of the expansion may be determined recursively by the following algorithm: If $n \geq 1$ and if the digits ϵ_i of the expansion of x are already defined for all $i < n$, then we let

$$\epsilon_n = 1 \text{ if } \sum_{i=1}^{n-1} \epsilon_i \lambda_i + \lambda_n < x. \tag{7a}$$

Otherwise, we set $\epsilon_n = 0$.

Then, each expansion with $x > 0$ is infinite, i.e. there is an infinite set of integers n with $\epsilon_n = 1$.

A proof of Theorem 1 can be found in [1], or in [7, exercise 131] or in [8].

For our purpose it is practical to introduce the following notion (see [6]):

Definition: A sequence (λ_n) satisfying conditions (4) and (5) of Theorem 1 is said to be interval-filling (relating to $[0, s]$) if every number $x \in [0, s]$ can be written in the form (6).

2. THE CASE $0 < r \leq 1$

First we give an example of an application of

Theorem 1: Let $\lambda_n = 1/F_n^r$ for all $n \in N$, where r is a fixed number with $0 < r \leq 1$. As we have mentioned above this sequence satisfies condition (4) of Theorem 1. (5) is also valid. For the proof of (7) we note first that $1/F_n < 2/F_{n+1}$ is valid for all $n \in N$. With $0 < r \leq 1$ we get

$$\frac{1}{F_n} < \frac{2}{F_{n+1}} \leq \frac{2^{1/r}}{F_{n+1}} \text{ which yields } \frac{1}{(F_n)^r} < \frac{2}{(F_{n+1})^r}.$$

From this we obtain by mathematical induction:

$$1/(F_n)^r - 1/(F_{n+k})^r < 1/(F_{n+1})^r + 1/(F_{n+2})^r + \dots + 1/(F_{n+k})^r$$

for all $k \geq 1$.

Now let $k \to \infty$. The limits of the two sides in the preceding inequality exist and we obtain

$$\frac{1}{(F_n)^r} \leq \sum_{k=1}^{\infty} \frac{1}{(F_{n+k})^r}.$$

Condition (7) is thus established. The application of Theorem 1 immediately yields, that each real number x with $0 < x \leq I_r$, where $0 < r \leq 1$, has (at least) one expansion of the form (1). In other words: $(1/F_n^r)_{n=1}^{\infty}$ is interval-filling relating to $[0, I_r]$.

This statement can be extended considerably.

Theorem 2: For each real number x with $0 < x < I_r$ and fixed r with $0 < r \leq 1$ the set C_x, which consists of all different expansions for x of the form (1), is uncountable; it has cardinality c (the power of the continuum).

The proof is based on an idea which is used in [2] and [3] considering the representation of the real number x in the form

$$x = \sum_{n=1}^{\infty} \epsilon_n q^{-n}$$

with non-integral base q. Such an expansion is not unique in general.

Our central point is the construction of a subsequence of $(1/(F_n)^r)_{n=1}^{\infty}$ which also satisfies the conditions of Theorem 1.

Before we give a proof of Theorem 2 we need some results on sums of Fibonacci reciprocals.

Theorem 3: (Jensen's inequality, see [5]). Let $0 < r \leq 1$ and let A be a finite or infinite subset of N. Then, we claim that

$$\sum_{i \in A} 1/F_i \leq \left(\sum_{i \in A} 1/(F_i)^r \right)^{1/r}.$$

Proof: Let us let $a = \left(\sum_{i \in A} 1/(F_i)^r\right)^{1/r}$. Thus, $\sum_{i \in A} 1/(F_i a)^r = 1$ and we get $1/(F_i a) \leq 1$ for all $i \in A$. $1 \geq r$ yields $1/(F_i a) \leq 1/(F_i a)^r$ for $i \in A$. Therefore,

$$\sum_{i \in A} 1/(F_i a) \leq \sum_{i \in A} 1/(F_i a)^r = 1.$$

Multiply by a. From the defintion of a and because of the last inequality we obtain thc assertion. □

Theorem 4: Let $0 < r \leq 1$. Let z denote a positive integer.

(i) If $z = 2k + 1$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r}.$$

(ii) If $z = 2k$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r} + \frac{1}{(F_{z+3})^r}.$$

(iii) If $z = 2k + 1$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+2})^r} + \frac{1}{(F_{z+3})^r} + \cdots + \frac{1}{(F_{z+n(z)})^r},$$

with an integer $n(z)$ dependent on the odd integer z, with $n(z) \leq n(z+2)$ and $n(2k+1) \to \infty$, as $k \to \infty$.

(iv) If $z = 2k$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+3})^r} + \frac{1}{(F_{z+4})^r} + \cdots + \frac{1}{(F_{z+k})^r}$$

with $k = 7$ if $z = 2$ and $k = 5$ if $z \geq 4$.

Proof: First we treat the case $r = 1$.

(i) $z = 2k + 1$. The assertion is equivalent to $F_{z+1}F_{z+2} < F_z(F_{z+1} + F_{z+2})$ or $F_{z+1}(F_{z+2} - F_z) < F_zF_{z+2}$ or $(F_{z+1})^2 < F_zF_{z+2}$. Then, the well-known formula $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$ with $n = z + 1$ yields $(F_{z+1})^2 = F_{z+2}F_z - 1 < F_{z+2}F_z$. The proof of (i) for $r = 1$ is complete.

(ii) $z = 2k$. Using (i), we get

$$\frac{1}{F_{z+1}} < \frac{1}{F_{z+2}} + \frac{1}{F_{z+3}} \text{ and then}$$

$$\frac{1}{F_z} < \frac{2}{F_{z+1}} < \frac{1}{F_{z+1}} + \frac{1}{F_{z+2}} + \frac{1}{F_{z+3}}.$$

(iii) $z = 2k + 1$. For the purpose of abbreviation let $\delta = \beta/\alpha = \left(\sqrt{5} - 3\right)/2$. Then, $|\delta| < 1$. Using the Binet's formula we have

$$\frac{F_z}{F_{z+2}} + \frac{F_z}{F_{z+3}} + \cdots + \frac{F_z}{F_{z+n}} = \frac{\alpha^z - \beta^z}{\alpha^{z+2} - \beta^{z+2}} + \cdots + \frac{\alpha^z - \beta^z}{\alpha^{z+n} - \beta^{z+n}}$$

$$= \alpha^{-2}\frac{(1 + |\delta|^z)}{1 + |\delta|^{z+2}} + \cdots + \alpha^{-n}\frac{(1 + |\delta|^z)}{1 \mp |\delta|^{z+n}}$$

$$> \frac{(1 + |\delta|^z)(1 - (/\alpha)^{n-1})}{(1 + |\delta|^{z+2})\alpha^2(1 - (1/\alpha))} \quad \text{(Note } \alpha^2(1 - (1/\alpha)) = 1!\text{)}$$

$$= \frac{(1 + |\delta|^z)(1 - (1/\alpha)^{n-1}}{(1 + |\delta|^{z+2})}.$$

Because $|\delta| < 1$ it follows that $(1 + |\delta|^z)/(1 + |\delta|^{z+2}) > 1$.

Further, we notice that the increasing sequence $((1 - (1/\alpha)^{n-1})$ has limit 1, as $n \to \infty$. Therefore, it follows that the inequality

$$\frac{(1 + |\delta|^z)(1 - (1/\alpha)^{n-1})}{(1 + |\delta|^{z+2})} > 1$$

is valid for all sufficient large values of $n \in N$. We denote the minimum of these values by $n(z)$. Thus, we have

$$\frac{F_z}{F_{z+2}} + \frac{F_z}{F_{z+3}} + \cdots + \frac{F_z}{F_{z+n}} > 1$$

for all $n \geq n(z)$. This is equivalent to (iii).

The assertions $n(z) \leq n(z+2)$ and $n(2k+1) \to \infty$ as $k \to \infty$ are easily proved.

(iv) Let $z = 2k$. If $z = 2$ a direct computation leads to the assertion. We observe that for $z \geq 4$, the desired result is equivalent to

$$\frac{F_z F_{z+1}}{F_{z-1}} \left(\frac{1}{F_{z+3}} + \frac{1}{F_{z+4}} + \frac{1}{F_{z+5}} \right) > 1.$$

Applying (i) with the odd integer $z+3$ to the parenthesis on the left hand side, we obtain

$$\frac{F_z F_{z+1}}{F_{z-1}} \left(\frac{1}{F_{z+3}} + \frac{1}{F_{z+4}} + \frac{1}{F_{z+5}} \right) > \frac{2F_z F_{z+1}}{F_{z-1} F_{z+3}}.$$

Therefore, it is enough to establish that $2F_z F_{z+1} > F_{z-1} F_{z+3}$. For that purpose we begin with the well-known equation $F_{n+2}F_{n-1} - F_n F_{n+1} = (-1)^n$ $(n \in N)$. We obtain $F_{z-2}F_{z+1} - F_{z-1}F_z = -1$ and from this $2F_{z+1}F_{z-2} > F_{z-1}F_z$. It follows step-by-step that $2F_{z+1}(F_z - F_{z-1}) > F_{z-1}F_z$; $2F_z F_{z+1} > 2F_{z-1}F_{z+1} + F_{z-1}F_z = F_{z-1}F_{z+3}$. We have therefore proved all parts of the theorem for $r = 1$.

The general assertions for $0 < r \leq 1$ are immediate consequences of Theorem 3. For instance: In the event of (i) the subset A is as follows: $A = \{z+1, z+2\}$. Then, we have by Theorem 3

$$\frac{1}{F_z} < \frac{1}{F_{z+1}} + \frac{1}{F_{z+2}} \leq \left(\frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r} \right)^{1/r}.$$

Raising both sides to the r^{th} power we have (i).

All other cases follow in a similar way. Therefore, the proof of Theorem 4 is complete. □

Before we continue with the proof of Theorem 2 let us give a simple application of Theorem 4.

r will be chosen with $0 < r \leq 1$. Assume that we have a representation of $x \in [0, I_r]$ in the form (1) with the interval-filling sequence $(1/(F_n)^r)_{n=1}^{\infty}$ on the basis of the algorithm (7a).

Theorem 5: Consider the sequence $(\epsilon_n(x))_{n=1}^{\infty}$ of digits. A chain of consecutive digits "1" following a digit "0" has at most length two.

Proof: Let $\epsilon_n(x) = 0, \epsilon_{n+1}(x) = 1, \epsilon_{n+2}(x) = 1, \ldots, \epsilon_{n+k}(x) = 1$ be a chain of the described kind. Then, we obtain by algorithm (7a)

$$\sum_{i=1}^{n-1} \frac{\epsilon_i}{(F_i)^r} + \frac{1}{(F_{n+1})^r} + \cdots + \frac{1}{(F_{n+k})^r} < x \leq \sum_{i=1}^{n-1} \frac{\epsilon_i}{(F_i)^r} + \frac{1}{(F_n)^r}.$$

Thus,

$$\frac{1}{(F_n)^r} > \frac{1}{(F_{n+1})^r} + \cdots + \frac{1}{(F_{n+k})^r}.$$

We now appeal to Theorem 4. It implies that k must be equal to 1 (at most equal to 2), if n is an odd (even) number since the assumption $k \geq 2$ ($k \geq 3$) leads to a contradiction with Theorem 4(i) or (ii).

The proof is complete. □

Proof of Theorem 2: We choose a sequence of even integers $(z_j)_{j=1}^{\infty} = (2k_j)_{j=1}^{\infty}$ with $z_{j+1} - z_j > \max\{9, n(z_j - 1)\}$ for all $j \in N$. The first member z_1 will be chosen (later) to be sufficiently large. Let $M = N - \{z_j\}_{j=1}^{\infty}$. Consider the set $\{1/(F_m)^r : m \in M\}$ as a non increasing sequence $(\lambda_n)_{n=1}^{\infty}$ of numbers: $\lambda_1 = 1/(F_1)^r, \lambda_2 = 1/(F_2)^r, \ldots, \lambda_{z_1-1} = 1/(F_{z_1-1})^r, \lambda_{z_1} = 1/(F_{z_1+1})^r, \lambda_{z_1+1} = 1/(F_{z_1+2})^r, \ldots$.

Next, we shall show that Theorem 1 is applicable to the sequence $(\lambda_n)_{n=1}^{\infty}$, in particular the validity of (6).

First we determine for each $m \in M$ the unique number $j \in N$ such that the condition $z_{j-1} + 1 \leq m \leq z_j - 1$ is satisfied ($z_o = 0$). Then, we obtain with the help of Theorem 4

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} + \frac{1}{(F_{m+3})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $z_{j-1} + 1 \leq m \leq z_j - 4$ in view of Theorem 4(i), (ii);

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $m = z_j - 3$ in view of Theorem 4(i);

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+3})^r} + \cdots + \frac{1}{(F_{m+k})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $m = z_j - 2$ in view of Theorem 4(iv); and

$$\sum_{n>m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+2})^r} + \frac{1}{(F_{m+3})^r} + \cdots + \frac{1}{(F_{m+n(m)})^r} > \frac{1}{(F_m)^r},$$

$n \in M$ if $m = z_j - 1$ in view of Theorem 4(iii).

So, we obtain for each $m \in M : 1/(F_m)^r < \sum_{n>m, n\in M} 1/(F_n)^r$, that is we proved that condition (7) of Theorem 1 is satisfied. It is clear that (4) and (5) are valid.

Let $0 < x < I_r$. We choose z_1 so that the following conditions are satisfied simultaneously:

$$(*) \quad \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} < x \text{ and } x + \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} < I_r.$$

This is possible, because $\lim_{z_1 \to \infty} \sum_{n \geq z_1} 1/(F_n)^r = 0$. Let $\triangle$ be any subset of the set $\{z_j\}_{j=1}^{\infty}$. We define now the 0-1-sequence $(\delta_j)_{j=1}^{\infty}$ in the following way: $\delta_j = 1$, if $z_j \in \triangle$, $\delta_j = 0$, if $z_j \notin \triangle$. Consider the number

$$y = x - \sum_{j=1}^{\infty} \frac{\delta_j}{(F_{z_j})^r}.$$

We obtain from the above conditions (*) that

$$y \geq x - \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} > 0 \text{ and } y \leq x < I_r - \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r}.$$

It follows from this that $0 < y < \sum_{n=1}^{\infty} \lambda_n$.

Now the key point is the application of Theorem 1. For each real number in the interval $[0, \sum_{n=1}^{\infty} \lambda_n]$ there is a series of the form

$$\sum_{n=1}^{\infty} \epsilon_n \lambda_n \quad \epsilon_n \in \{0, 1\}.$$

With a view to the definition of y we receive the following representation:

$$x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n + \sum_{j=1}^{\infty} \frac{\delta_j}{(F_{z_j})^r}.$$

We note that a Fibonacci reciprocal contained in the second sum cannot occur in the first, which implies that the representation of x is dependent on the sequence (δ_j). Two different sequence (δ_j) and (δ'_j) lead to different representations of x. It is well-known that the set of all 0-1-sequences has cardinality c (the power of the continuum). Therefore, the set C_x of different representations of x in the form (1) has at least cardinality c. Because the cardinality of the set of 0-1-sequences equals the cardinality of the continuum, the set C_x has cardinality at most c.

Theorem 2 is thus established. □

Next, we will draw a comparison between our Theorem 2 and results in [2] and [3], which are due to P. Erdös, M. Horvath, I. Joó and V. Komornik.

First we make the observation that by Binet's formula $\lim_{n \to \infty} F_n/\alpha^n = 1/\sqrt{5}$, that is F_n and α^n are "almost" proportional as $n \to \infty$. To simplify matters we assume $F_n \sim \alpha^n$. Then it follows that $(F_n)^r \sim \alpha^{nr} = q^n$ with $q = \alpha^r$ and the interval $0 < r \leq 1$ corresponds to the interval $1 < q \leq \alpha$.

Let $q \in (1, \alpha)$. It was proven in [2] (see Theorem 3 in [2]) that for every $x \in (0, 1/q - 1)$ there are c different expansions of the form

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{q^n} \quad \epsilon_n \in \{0, 1\}. \tag{8}$$

We can say that this result is analogous to our Theorem 2, if we take into consideration the above-mentioned remark on $(F_n)^r$ and q^n.

On the other hand it was shown in [3] (see the proof of Theorem 1 in [3]) that, if we assume in (8) $x = 1$ and $q = \alpha$, there exist precisely countably many expansions of the form (8). It is surprising that we have different cardinal numbers relating the set of representations for $x = 1$ and $r = 1$ according to (1) and the set of representation for $x = 1$ and $q = \alpha$ according to (8).

3. THE CASE $r > 1$

We shall prove two further theorems regarding expansions of the form (1).

Theorem 6: Let r satisfy $1 < r < \log 2/\log\alpha$. Then, there is an even integer $m(r)$ such that the sequence $(1/(F_n)^r)_{n=m(r)-1}^{\infty}$ is interval-filling.

Theorem 7: Let r satisfy $r \geq \log 2/\log\alpha$. Then, there is no integer $m \in N$ such that $(1/(F_n)^r)_m^\infty$ is an interval-filling sequence.

Proof of Theorem 6: In view of the equation $\beta/\alpha = -1/\alpha^2$ and with the help of Binet's formula it easily follows that

$$F_{n+1}/F_n = \alpha E(n) \text{ where } E(n) = \frac{1+(-1)^{n+2}\alpha^{-2n-2}}{1+(-1)^{n+1}\alpha^{-2n}}.$$

If $1 < r < \log 2/\log\alpha$ holds, then $2 > 2^{1/r} > \alpha$. As soon as n is an odd integer we get $E(n) < 1$. Thus, it follows that $F_{n+1}/F_n < 2^{1/r}$ for an odd integer n. On the other hand, we obtain from the definition of $E(n)$ that for even integers the following statements are valid: $E(n) > 1, E(n) > E(n+2), \lim_{n\to\infty} E(n) = 1$. Hence, there is a smallest even integer $m(r)$ such that $1 < E(m(r)) < 2^{1/r}/\alpha$. Therefore, $F_{n+1}/F_n = \alpha E(n) \leq \alpha E(m(r)) < 2^{1/r}$ for each even $n \geq m(r)$. Summarizing we obtain $F_{n+1}/F_n < 2^{1/r}$ or $(F_{n+1})^r/(F_n)^r < 2$ for all integers $n \geq m(r) - 1$. This implies that Theorem 1 is applicable since the sequence $(1/(F_n)^r)$ with $n \geq m(r) - 1$ meets all the requirements of the theorem, in particular condition (7). Theorem 6 is thus established. □

Proof of Theorem 7: Let $r \geq \log 2/\log\alpha$, equivalent to $\alpha^r \geq 2$. First we shall prove that, for all even integers $n \in N$, we have

$$1/(F_n)^r > 1/(F_{n+1})^r + 2/(F_{n+2})^r. \tag{10}$$

We again use the defintion of $E(n)$ in the proof of Theorem 6. We receive from (10) the equivalent inequality

$$\alpha^r(E(n)E(n+1))^r > (E(n+1))^r + 2/\alpha^r \ (n \text{ even}). \tag{11}$$

On the other side, we obtain for even $n \in N$:

$$E(n)E(n+1) = 1 + \frac{1-1/\alpha^{+4}}{\alpha^{2n}-1} > 1 \text{ and } E(n+1) < 1.$$

The last two inequalities and $\alpha^r \geq 2$ yield that (11) is valid for all even $n \in N$, because

$$\alpha^r(E(n)E(n+1))^r \geq 2((E(n)E(n+1))^r > 2 > (E(n+1))^r + 1 \geq (E(n+1))^r + 2/\alpha^r.$$

Thus, the equivalent statement (10) follows, from which we obtain by mathematical induction:

$$\frac{1}{(F_n)^r} - \frac{1}{(F_{n+2k})^r} > \sum_{i=1}^{2k} \frac{1}{(F_{n+i})^r} \tag{12}$$

for all $k \geq 1$ and even n.

Then, it follows from (12) as $k \to \infty$:

$$\frac{1}{(F_n)^r} \geq \sum_{i=1}^{\infty} \frac{1}{(F_{n+i})^r} \quad (n \in N,\ n \text{ even}). \tag{13}$$

Now, suppose that in (13) for two consecutive even numbers $n = v$ and $n = v + 2$ the equals sign is valid.

Then, a simple calculation shows that we have a contradiction to (10):

$$\frac{1}{(F_v)^r} = \frac{1}{(F_{v+1})^r} + \frac{2}{(F_{v+2})^r},$$

i.e. from two successive inequalities (13) there is at most one equality. Next, consider the set

$$A(r) - \{n | n \in 2N,\ 1/(F_n)^r > \sum_{i=1}^{\infty} 1/(F_{n+i})^r\}.$$

From the preceding argument it is clear that $A(r)$ is an infinite subset of N, such that condition (7) of Theorem 1 is not true for $n \in A(r)$. We conclude that there is no integer $m \in N$, such that the sequence $(1/(F_n)^r)_{n=m}^{\infty}$ is interval-filling. □

REFERENCES

[1] J. L. Brown, Jr. "On Generalized Bases for Real Numbers." *The Fibonacci Quarterly* **9** (1971): 477-496.

[2] P. Erdös, I. Joo and V. Komornik. "Characterisation of the Unique Expansions $1 = \sum q^{-n}i$ and Related Problems." *Bull. Soc. Math. France* **118** (1990): 377-390.

[3] P. Erdös, M. Horvath and I. Joo. "On the Uniqueness of the Expansion $1 = \sum q^{-n_i}$." *Acta Math. Hung.* **58** (1991): 333-342.

[4] P. Erdös, J. Joo and I. Joo. "On a Problem of Tamas Varga." *Bull. Soc. Math. Fr.* **120** (1992): 507-521.

[5] F. Hirzebruch, W. Scharlau. *Einführung in Die Funktionalanalysis*, Bibliographisches Institut, Mannheim, 1971.

[6] J.C. Parnami. "Interval-filling Sequences and Completely Additive Functions." *Acta Math. Hung.* **58** (1991): 229-240.

[7] G. Polya und G. Szegö. *Aufgaben und Lehrsätze aus der Analysis I*, 4. Auflage, Springer-Verlag, 1970.

[8] P. Ribenboim. "Representations of Real Numbers by Means of Fibonacci Numbers." *L.' Enseignement Mathématique* **31** (1985): 249-259.

[9] S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section, Theory and Applications.* E. Horwood, Chichester, 1989.

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