THE LINEAR ALGEBRA OF THE GENERALIZED FIBONACCI MATRICES

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1. INTRODUCTION

Let x be any nonzero real number. The n by n *generalized Fibonacci matrix of the first kind*, $\mathcal{F}_n[x] - [f_{ij}]$, is defined as

$$f_{ij} = \begin{cases} F_{i-j+1}x^{i-j} & i-j+1 \geq 0, \\ 0 & i-j+1 < 0. \end{cases} \tag{1}$$

We define the n by n *generalized Fibonacci matrix of the second kind*, $\mathcal{R}_n[x] = [r_{ij}]$, as

$$r_{ij} = \begin{cases} F_{i-j+1}x^{i+j-2} & i-j+1 \geq 0, \\ 0 & i-j+1 < 0. \end{cases} \tag{2}$$

Note that $\mathcal{F}_n[1] = \mathcal{R}_n[1]$ and $\mathcal{F}_n[1]$ is called the Fibonacci matrix (see [3]).

The n by n *generalized symmetric Fibonacci matrix*, $\mathcal{Q}_n[x] = [q_{ij}]$, is defined as

$$q_{ij} = q_{ji} = \begin{cases} \sum_{k=1}^{i} F_k^2 x^{2i-2} & i - j, \\ q_{i,j-2}x^2 + q_{i,j-1}x & i+1 \leq j, \end{cases}$$

where $q_{1,0} = 0$. Then we know that for $j \geq 1$, $q_{1j} = q_{j1} = F_j x^{j-1}$ and $q_{2j} = q_{j2} = F_{j+1}x^j$. $\mathcal{Q}_n[1]$ is called the symmetric Fibonacci matrix (see [3]). For example,

$$\mathcal{F}_5[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ 2x^2 & x & 1 & 0 & 0 \\ 3x^3 & 2x^2 & x & 1 & 0 \\ 5x^4 & 3x^3 & 2x^2 & x & 1 \end{bmatrix}, \quad \mathcal{R}_5[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 & 0 \\ 2x^2 & x^3 & x^4 & 0 & 0 \\ 3x^3 & 2x^4 & x^5 & x^6 & 0 \\ 5x^4 & 3x^5 & 2x^6 & x^7 & x^8 \end{bmatrix}$$

$$\mathcal{Q}_5[x] = \begin{bmatrix} 1 & x & 2x^2 & 3x^3 & 5x^4 \\ x & 2x^2 & 3x^3 & 5x^4 & 8x^5 \\ 2x^2 & 3x^3 & 6x^4 & 9x^5 & 15x^6 \\ 3x^3 & 5x^4 & 9x^5 & 15x^6 & 24x^7 \\ 5x^4 & 8x^5 & 15x^6 & 24x^7 & 40x^8 \end{bmatrix}.$$

Let $\mathcal{D} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in R^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$, where R is the set of real numbers. For $\boldsymbol{x},\ \boldsymbol{y} \in \mathcal{D},\ \boldsymbol{x} \prec \boldsymbol{y}$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i,\ k = 1, 2, \ldots, n$ and if $k = n$ then equality holds. When $\boldsymbol{x} \prec \boldsymbol{y}$, $\boldsymbol{x}$ is said to be *majorized* by $\boldsymbol{y}$, or $\boldsymbol{y}$ is said to *majorize* $\boldsymbol{x}$. The condition for majorization can be rewritten as follows: for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}, \boldsymbol{x} \prec \boldsymbol{y}$ if $\sum_{i=0}^{k} x_{n-i} \geq \sum_{i=0}^{k} y_{n-i},\ k = 0, 1, \ldots, n-2$ and if $k = n-1$ then equality holds.

The following is an interesting simple fact.

$$(\overline{x}, \ldots, \overline{x}) \prec (x_1, \ldots, x_n),$$

where $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$. More interesting facts about majorization can be found in [4].

An $n \times n$ matrix $P = [p_{ij}]$ is *doubly stochastic* if $p_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n, \sum_{i=1}^{n} p_{ij} = 1, j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} p_{ij} = 1, i = 1, 2, \ldots, n$. In 1929, Hardy, Littlewood and Polya proved that a necessary and sufficient condition that $\boldsymbol{x} \prec \boldsymbol{y}$ is that there exist a doubly stochastic matrix P such that $\boldsymbol{x} = \boldsymbol{y}P$.

We know both the eigenvalues and the main diagonal elements of a real symmetric matrix, are real numbers. The precise relationship between the main diagonal elements and the eigenvalues is given by the notion of majorization as follows: the vector of eigenvalues of a symmetric matrix majorize the main diagonal elements of the matrix (see [2]).

In [1] and [5], the authors gave factorizations of the Pascal matrix and generalized Pascal matrix. In [3],the authors gave factorizations of the Fibonacci matrix $\mathcal{F}_n[1]$ and discussed the Cholesky factorization and the eigenvalues of the symmetric Fibonacci matrix $\mathcal{Q}_n[1]$.

In this paper, we consider factorizations of the generalized Fibonacci matrices of the first kind and the second kind, and consider the Cholesky factorization of the generalized symmetric Fibonacci matrix. Also, we consider the eigenvalues of $\mathcal{Q}_n[x]$.

2. FACTORIZATIONS

In this section, we discuss factorizations of $\mathcal{F}_n[x]$, $\mathcal{R}_n[x]$ and $\mathcal{Q}_n[x]$ for any nonzero real number x.

Let I_n be the identity matrix of order n. We define the matrices $S_n[x]$, $\overline{\mathcal{F}}_n[x]$ and $G_k[x]$ by

$$S_0[x] = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & 0 & 1 \end{bmatrix}, S_{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{bmatrix},$$

and $S_k[x] = S_0[x] \oplus I_k,\ k = 1, 2, \ldots,\ \overline{\mathcal{F}}_n[x] = [1] \oplus \mathcal{F}_{n-1}[x],\ G_1[x] = I_n,\ G_2[x] = I_{n-3} \oplus S_{-1}[x]$, and, for $k \geq 3,\ G_k[x] = I_{n-k} \oplus S_{k-3}[x]$.

In [3], the authors gave a factorization of the Fibonacci matrix $\mathcal{F}_n[1]$ as follows:

Theorem 2.1: For $n \geq 1$ a positive integer,

$$\mathcal{F}_n[1] = G_1[1]G_2[1] \ldots G_n[1].$$

Now, we consider a factorization of the generalized Fibonacci matrix of the first kind. From the definition of the matrix product and a familiar Fibonacci sequence, we have the following lemma.

Lemma 2.2: For $k \geq 3$,

$$\overline{\mathcal{F}}_k[x]S_{k-3}[x] = \mathcal{F}_k[x].$$

Recall that $G_n[x] = S_{n-3}[x]$, $G_1[x] = I_n$ and $G_2[x] = I_{n-3} \oplus S_{-1}[x]$. As an immediate consequence of lemma 2.2, we have the following theorem.

Theorem 2.3: The n by n generalized Fibonacci matrix of the first kind, $\mathcal{F}_n[x]$, can be factorized by $G_k[x]$'s as follows.

$$\mathcal{F}_n[x] = G_1[x]G_2[x]\dots G_n[x].$$

We consider another factorization of $\mathcal{F}_n[x]$. Then n by n matrix $C_n[x] = [c_{ij}]$ is defined as:

$$c_{ij} = \begin{cases} F_i x^{i-j} & j = 1, \\ 1 & i = j, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{i.e., } C_n[x] = \begin{bmatrix} F_1 & 0 & \dots & 0 \\ F_2 x & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n x^{n-1} & 0 & \dots & 1 \end{bmatrix}.$$

The next theorem follows, by a simple calculation.

Theorem 2.4: For $n \geq 2$,

$$\mathcal{F}_n[x] = C_n[x](I_1 - \oplus C_{n-1}[x])(I_2 \oplus C_{n-2}[x])\dots(I_{n-2} \oplus C_2[x]).$$

Also we can easily find the inverse of the generalized Fibonacci matrix of the first kind. We know that

$$S_0[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ -x^2 & 0 & 1 \end{bmatrix}, \quad S_{-1}[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x & 1 \end{bmatrix},$$

and $S_k[x]^{-1} = S_0[x]^{-1} \oplus I_k$. Define $H_k[x] = G_k[x]^{-1}$. Then $H_1[x] = G_1[x]^{-1} = I_n$, $H_2[x] = G_2[x]^{-1} = I_{n-3} \oplus S_{-1}[x]^{-1} = I_{n-2} \oplus \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}$ and $H_n[x] = S_{n-3}[x]^{-1}$. Also, we know that

$$C_n[x]^{-1} = \begin{bmatrix} F_1 & 0 & \dots & 0 \\ -F_2 x & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -F_n x^{n-1} & 0 & \dots & 1 \end{bmatrix} \text{ and } (I_k \oplus C_{n-k}[x])^{-1} = I_k \oplus C_{n-k}[x]^{-1}.$$

So, the following corollary holds.

Corollary 2.5: For $n \geq 2$,

$$\begin{aligned} \mathcal{F}_n[x]^{-1} &= G_n[x]^{-1}G_{n-1}[x]^{-1}\dots G_2[x]^{-1}G_1[x]^{-1} \\ &= H_n[x]H_{n-1}[x]\dots H_2[x]H_1[x] \\ &= (I_{n-2} \oplus C_2[x]^{-1})\dots(I_1 \oplus C_{n-1}[x]^{-1})C_n[x]^{-1}. \end{aligned}$$

From corollary 2.5, we have

$$\mathcal{F}_n[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -x & 1 & 0 & 0 & \dots & 0 \\ -x^2 & -x & 1 & 0 & \dots & 0 \\ 0 & -x^2 & -x & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -x^2 & -x & 1 \end{bmatrix}. \tag{3}$$

For a factorization of the generalized Fibonacci matrix of the second kind, $\mathcal{R}_n[x]$, we define the matrices $M_n[x]$, $\overline{\mathcal{R}}_n[x]$ and $N_k[x]$ by

$$M_0[x] = \begin{bmatrix} 1 & 0 & 0 \\ x & x^2 & 0 \\ 1 & 0 & x^2 \end{bmatrix}, \quad M_{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & x^2 \end{bmatrix},$$

and $M_k[x] = M_0[x] \oplus x^2 I_k, k = 1, 2, \dots, \overline{\mathcal{R}}_n[x] = [1] \oplus \mathcal{R}_{n-1}[x], N_1[x] = I_n, N_2[x] = I_{n-3} \oplus M_{-1}[x]$, and, for $k \geq 3$, $N_k[x] = I_{n-k} \oplus M_{k-3}[x]$. Then we have the following lemma.

Lemma 2.6: For $k \geq 3$,

$$\mathcal{R}_k[x] = \overline{\mathcal{R}}_k[x] M_{k-3}[x].$$

Proof: For $k = 3$, we have $\overline{\mathcal{R}}_3[x] M_0[x] = \mathcal{R}_3[x]$. Let $k > 3$. From the definition of the matrix product and a familiar Fibonacci sequence, the conclusion follows. □

As an immediate consequence of lemma 2.6, we have the following theorem.

Theorem 2.7: The n by n generalized Fibonacci matrix of the second kind, $\mathcal{R}_n[x]$, can be factorized by N_k's as follows.

$$\mathcal{R}_n[x] = N_1[x] N_2[x] \dots N_n[x].$$

Now, we consider another factorization of $\mathcal{R}_n[x]$. The n by n matrix $L_n[x] = [l_{ij}]$ is defined as:

$$l_{ij} = \begin{cases} F_i x^{i-j} & j = 1, \\ x^2 & i = j, j \geq 2 \\ 0 & \text{otherwise,} \end{cases} \qquad \text{i.e., } L_n[x] = \begin{bmatrix} F_1 & 0 & \dots & 0 \\ F_2 x & x^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n x^{n-1} & 0 & \dots & x^2 \end{bmatrix}.$$

From the definition of the matrix $L_n[x]$, the following theorem holds.

Theorem 2.8: For $n \geq 2$,

$$\mathcal{R}_n[x] = L_n[x](I_1 \oplus L_{n-1}[x])(I_2 \oplus L_{n-2}[x]) \dots (I_{n-2} \oplus L_2[x]).$$

We can easily find the inverse of the generalized Fibonacci matrix of the second kind. We know that

$$M_0^{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{x} & \frac{1}{x^2} & 0 \\ -\frac{1}{x^2} & 0 & \frac{1}{x^2} \end{bmatrix}, \quad M_{-1}^{-1}[x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{x} & \frac{1}{x^2} \end{bmatrix},$$

and for $k \geq 1$,

$$M_k^{-1}[x] = M_0^{-1}[x] \oplus \frac{1}{x^2} I_k.$$

Define $U_k[x] = N_k^{-1}[x]$. Then $U_1[x] = I_n$, $U_2[x] = N_2^{-1}[x] = I_{n-3} \oplus M_{-1}^{-1}[x]$, and, for $k \geq 3$, $U_k[x] = N_k^{-1}[x] = I_{n-k} \oplus M_{k-3}^{-1}[x]$. Also, we know that

$$L_n[x]^{-1} = \begin{bmatrix} F_1 & 0 & \dots & \dots & \dots & 0 \\ -\frac{F_2}{x} & \frac{1}{x^2} & \dots & \dots & \dots & 0 \\ -F_3 & 0 & \frac{1}{x^2} & \dots & \dots & 0 \\ -F_4 x & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -F_n x^{n-3} & 0 & \dots & 0 & 0 & \frac{1}{x^2} \end{bmatrix}$$

and $(I_k \oplus L_{n-k}[x])^{-1} = I_k \oplus L_{n-k}[x]^{-1}$. Then we have the following corollary.

Corollary 2.9: For $n \geq 2$,

$$\begin{aligned} \mathcal{R}_n[x]^{-1} &= U_n[x] U_{n-1}[x] \dots U_1[x] \\ &= (I_{n-2} \oplus L_2[x]^{-1}) \dots (I_1 \oplus L_{n-1}[x]^{-1}) L_n[x]^{-1}. \end{aligned}$$

From corollary 2.9, we have

$$\mathcal{R}_n[x]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{x} & \frac{1}{x^2} & 0 & 0 & \dots & 0 \\ -\frac{1}{x^2} & -\frac{1}{x^3} & \frac{1}{x^4} & 0 & \dots & 0 \\ 0 & -\frac{1}{x^4} & -\frac{1}{x^5} & \frac{1}{x^6} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{1}{x^{2n-4}} & -\frac{1}{x^{2n-3}} & \frac{1}{x^{2n-2}} \end{bmatrix}. \tag{4}$$

Note that $\mathcal{F}_n[1]^{-1} = \mathcal{R}_n[1]^{-1}$.

Now, we consider a factorization of $\mathcal{Q}_n[x]$. In [3], the authors gave the Cholesky factorization of the symmetric Fibonacci matrix $\mathcal{Q}_n[1]$ as follows:

Theorem 2.10: For $n \geq 1$ a positive integer,

$$\mathcal{Q}_n[1] = \mathcal{F}_n[1] \mathcal{F}_n[1]^T.$$

From the definition of $\mathcal{Q}_n[x]$, we derive the following lemma.

Lemma 2.11: For $n \geq 1$ a positive integer, let $\mathcal{Q}_n[x] = [q_{ij}]$. Then

(i) For $j \geq 3$, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)x^{j+1}$.

(ii) For $j \geq 4$, $q_{4j} = F_4(F_{j-4} + F_{j-4}F_3 + F_{j-3}F_5)x^{j+2}$.

(iii) For $j \geq 5$, $q_{5j} = [F_{j-5}F_4(1 + F_3 + F_5) + F_{j-4}F_5F_6]x^{j+3}$.

(iv) For $j \geq i \geq 6$, $q_{ij} = [F_{j-i}F_4(1 + F_3 + F_5) + F_{j-i}F_5F_6 + \cdots + F_{j-i}F_{i-1}F_i + F_{j-i+1}F_iF_{i+1}]x^{i+j-2}$.

Proof: We know that $q_{3,3} = \sum_{k=1}^{3} F_k^2 x^4 = (F_1^2 + F_2^2 + F_3^2)x^4 = F_3F_4x^4$, and hence $q_{3,3} = F_4F_3x^4 = F_4(F_0 + F_1F_3)x^4$ for $F_0 = 0$. By induction, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)x^{j+1}$ for $j \geq 3$. Thus, we have (i).

We know that $q_{1,3} = q_{3,1} = F_3x^2$ and $q_{2,3} = q_{3,2} = F_4x^3$. Also, we know that $q_{4,1} = q_{1,4} = F_4x^3, q_{4,2} = q_{2,4} = F_5x^4$ and $q_{3,4} = q_{4,3} = F_4(F_1 + F_2F_3)x^5$ by (i). By induction, we have $q_{4j} = F_4(F_{j-4} + F_{j-4}F_3 + F_{j-3}F_5)x^{j+2}$ for $j \geq 4$. Thus, (ii) holds.

By induction, (iii) and (iv) also hold. □

Now, we have the following theorem.

Theorem 2.12: For $n \geq 1$ a positive integer

$$U_n[x]U_{n-1}[x] \ldots U_1[x]\mathcal{Q}_n[x] = \mathcal{F}_n[x]^T$$

and the Cholesky factorization of $\mathcal{Q}_n[x]$ is given by

$$\mathcal{Q}_n[x] = \mathcal{R}_n[x]\mathcal{F}_n[x]^T.$$

Proof: By corollary 2.9, $U_n[x]U_{n-1}[x] \ldots U_1[x] = \mathcal{R}_n[x]^{-1}$. So, if we have $\mathcal{R}_n[x]^{-1}\mathcal{Q}_n[x] = \mathcal{F}_n[x]^T$ then the theorem holds.

Note that $\mathcal{Q}_n[x]$ is a symmetric matrix. Let $A[x] = [a_{ij}] = \mathcal{R}_n[x]^{-1}\mathcal{Q}_n[x]$. By the definition of $\mathcal{Q}_n[x]$ and (4), $a_{ij} = 0$ for $j + 1 \leq i$.

Now we consider the case $j \geq i$. By (4) and lemma 2.11, we know that $a_{ij} = f_{ji}$ for $i \leq 5$.

We consider $j \geq i \geq 6$. Then, by (4), we have

$$
\begin{aligned}
a_{ij} &= -\frac{1}{x^{2i-4}}q_{i-2,j} - \frac{1}{x^{2i-3}}q_{i-1,j} + \frac{1}{x^{2i-2}}q_{i,j} \\
&= \frac{1}{x^{2i-2}}[F_{j-i}F_4(1+F_3+F_5) + F_{j-1}F_5F_6 + \cdots + F_{j-1}F_{i-1}F_i \\
&\quad + F_{j-i+1}F_iF_{i+1}]x^{i+j-2} \\
&- \frac{1}{x^{2i-3}}[F_{j-i+1}F_4(1+F_3+F_5) + F_{j-i+1}F_5F_6 + \cdots + \\
&\quad F_{j-i+1}F_{i-2}F_{i-1} + F_{j-i+2}F_{i-1}F_i]x^{i+j-3} \\
&- \frac{1}{x^{2i-4}}[F_{j-i+2}F_4(1+F_3+F_5) + F_{j-i+2}F_5F_6 + \cdots + \\
&\quad F_{j-i+2}F_{i-3}F_{i-2} + F_{j-i+3}F_{i-2}F_{i-1}]x^{i+j-4} \\
&= [(F_{j-i} - F_{j-i+1} - F_{j-i+2})F_4(1+F_3+F_5) + (F_{j-i} - F_{j-i+1} \\
&\quad - F_{j-i+2})F_5F_6 + \cdots + (F_{j-i} - F_{j-i+1} - F_{j-i+2})F_{i-3}F_{i-2} \\
&\quad + (F_{j-i} - F_{j-i+1} - F_{j-i+3})F_{i-2}F_{i-1} \\
&\quad + (F_{j-i} - F_{j-i+2})F_{i-1}F_i + F_{j-i+1}F_iF_{i+1}]x^{j-i}.
\end{aligned}
$$

Since $F_{j-i} - F_{j-i+1} - F_{j-i+2} = -2F_{j-i+1}, F_{j-i} - F_{j-i+1} - F_{j-i+3} = -3F_{j-i+1}$, and $F_{j-i} - F_{j-i+2} = -F_{j-i+1}$, we have

$$a_{ij} = F_{j-i+1}[-2F_4 - 2(F_3F_4 + F_4F_5 + \cdots + F_{i-2}F_{i-1}) - F_{i-2}F_{i-1} - F_{i-1}F_i + F_iF_{i+1}]x^{j-i}.$$

Since $F_4 = 3$ and

$$F_1F_2 + F_2F_3 + \cdots + F_{i-1}F_i = \frac{F_{2i-1} + F_iF_{i-1} - 1}{2},$$

we have

$$
\begin{aligned}
a_{ij} &= \left[-6 - 2\left(\frac{F_{2(i-1)-1} + F_{i-1}F_{(i-1)-1} - 1}{2} - F_1F_2 - F_2F_3\right)\right. \\
&- \quad F_{i-2}F_{i-1} - F_{i-1}F_i + F_iF_{i+1}]F_{j-i+1}x^{j-i} \\
&= (1 - 2F_{i-1}F_{i-2} - F_{2i-3} - F_{i-1}F_i + F_iF_{i+1})F_{j-i+1}x^{j-i}.
\end{aligned}
$$

Since $F_{i+1} = F_i + F_{i-1}$ and $F_{i+1}^2 + F_i^2 = F_{2i+1}$,

$$\begin{aligned} a_{ij} &= (1 - 2F_{i-1}F_{i-2} - (F_{i-1}^2 + F_{i-2}^2) + F_i^2) + F_{j-i+1}x^{j-i} \\ &= F_{j-i+1}x^{j-i} \\ &= f_{ji}. \end{aligned}$$

Thus, $A[x] = \mathcal{F}_n[x]^T$ for $1 \le i,\ j \le n$.

Therefore, $\mathcal{R}_n[x]^{-1}\mathcal{Q}_n[x] = \mathcal{F}_n[x]^T$, i.e., the Cholesky factorization of $\mathcal{Q}_n[x]$ is given by $\mathcal{Q}_n[x] = \mathcal{R}_n[x]\mathcal{F}_n[x]^T$. □

For example,

$$\mathcal{Q}_5[x] = \begin{bmatrix} 1 & x & 2x^2 & 3x^3 & 5x^4 \\ x & 2x^2 & 3x^3 & 5x^4 & 8x^5 \\ 2x^2 & 3x^3 & 6x^4 & 9x^5 & 15x^6 \\ 3x^3 & 5x^4 & 9x^5 & 15x^6 & 24x^7 \\ 5x^4 & 8x^5 & 15x^6 & 24x^7 & 40x^8 \end{bmatrix}$$

$$= \mathcal{R}_5[x]\mathcal{F}_5[x]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 & 0 \\ 2x^2 & x^3 & x^4 & 0 & 0 \\ 3x^3 & 2x^4 & x^5 & x^6 & 0 \\ 5x^4 & 3x^5 & 2x^6 & x^7 & x^8 \end{bmatrix} \begin{bmatrix} 1 & x & 2x^2 & 3x^3 & 5x^4 \\ 0 & 1 & x & 2x^2 & 3x^3 \\ 0 & 0 & 1 & x & 2x^2 \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\mathcal{Q}_n[x]^{-1} = (\mathcal{F}_n[x]^T)^{-1}\mathcal{R}_n[x]^{-1}$, we have

$$\mathcal{Q}_n[x]^{-1} = \begin{bmatrix} 3 & 0 & -\frac{1}{x^2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{x^2} & 0 & -\frac{1}{x^4} & 0 & 0 & \cdots & 0 \\ -\frac{1}{x^2} & 0 & \frac{3}{x^4} & 0 & -\frac{1}{x^6} & 0 & \cdots & 0 \\ 0 & -\frac{1}{x^4} & 0 & \frac{3}{x^6} & 0 & -\frac{1}{x^8} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{x^{2n-8}} & 0 & \frac{3}{x^{2n-6}} & 0 & -\frac{1}{x^{2n-4}} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{x^{2n-6}} & 0 & \frac{2}{x^{2n-4}} & -\frac{1}{x^{2n-3}} \\ 0 & \cdots & \cdots & \cdots & 0 & -\frac{1}{x^{2n-4}} & -\frac{1}{x^{2n-3}} & \frac{1}{x^{2n-2}} \end{bmatrix}. \quad (5)$$

From theorem 2.12, we have the following corollary.

Corollary 2.13: If k is an odd number, then

$$(F_nF_{n-k} + \cdots + F_{k+1}F_1)x^{2n-k-2} = \begin{cases} (F_nF_{n-(k-1)} - F_k)x^{2n-k-2} & \text{if } n \text{ is odd,} \\ (F_nF_{n-(k-1)})x^{2n-k-2} & \text{if } n \text{ is even.} \end{cases}$$

If k is an even number, then

$$(F_nF_{n-k}+\cdots+F_{k+1}F_1)x^{2n-k-2}=\begin{cases}(F_nF_{n-(k-1)})x^{2n-k-2} & \text{if } n \text{ is odd,}\\ (F_nF_{n-(k-1)}-F_k)x^{2n-k-2} & \text{if } n \text{ is even.}\end{cases}$$

3. EIGENVALUES OF $\mathcal{Q}_n[x]$

Let A be an m by n matrix. For index sets $\alpha \subseteq \{1,2,\ldots,m\}$ and $\beta \subseteq \{1,2,\ldots,n\}$, we denote the submatrix that lies in the rows of A indexed by α and the columns indexed by β as $A(\alpha,\beta)$. If $m=n$ and $\alpha=\beta$, the submatrix $A(\alpha,\alpha)$ is a principal submatrix of A and is abbreviated $A(\alpha)$. We denote by A_i the leading principal submatrix of A determined by the first i rows and columns, $A_i \equiv A(\{1,2,\ldots,i\}), i=2,\ldots,n$. Note that if A is Hermitian, so is each A_i, and therefore each A_i has a real determinant.

We know that if A is positive definite, then all principal minors of A are positive, and, in fact, the converse is valid when A is Hermitian. However, in [2], we have the following stronger result: If A is an n by n Hermitian matrix, then A is positive definite if and only if $\det A_i > 0$ for $i=1,2,\ldots,n$. We know that $\mathcal{Q}_n[x]$ is a Hermitian matrix, $\det \mathcal{R}_n[x]=x^{n(n-1)}$ and $\det \mathcal{F}_n[x]=1$ for $n\geq 2$. By theorem 2.12, we have $\det \mathcal{Q}_n[x]=\det(\mathcal{R}_n[x]\mathcal{F}_n[x]^T)=x^{n(n-1)}$. Since x is a nonzero real, we have $\det \mathcal{Q}_i[x]>0, i=2,3,\ldots,n$. Thus, the matrix $\mathcal{Q}_n[x]$ is a positive definite matrix, and hence the eigenvalues of $\mathcal{Q}_n[x]$ are all positive.

Let $\lambda_1[x],\lambda_2[x],\ldots,\lambda_n[x]$ be the eigenvalues of $\mathcal{Q}_n[x]$. Since

$$q_{ii}=\sum_{k=1}^{i}F_k^2x^{2i-2}=F_{i+1}F_ix^{2i-2},$$

we have

$$(F_{n+1}F_nx^{2n-2},F_nF_{n-1}x^{2n-4},\ldots,F_3F_2x^2,F_2F_1)\prec(\lambda_1[x],\lambda_2[x],\ldots,\lambda_n[x]).$$

Let $s_n[x]=\sum_{i=1}^{n}\lambda_i[x]$. Then,

$$s_n[x]=F_{n+1}F_nx^{2n-2}+F_nF_{n-1}x^{2n-4}+\cdots+F_3F_2x^2+F_2F_1.$$

Thus, $\lambda_1[1],\lambda_2[1],\ldots,\lambda_n[1]$ are the eigenvalues of $\mathcal{Q}_n[1]$ and

$$(F_{n+1}F_n,F_nF_{n-1},\ldots,F_3F_2,F_2F_1)\prec(\lambda_1[1],\lambda_2[1],\ldots,\lambda_n[1]).$$

We know the interesting combinatorial property

$$\sum_{i=0}^{n}\binom{n-i}{i}=F_{n+1}.$$

In [3], the authors gave the following result:

$$\lambda_1[1]+\lambda_2[1]+\cdots+\lambda_n[1]=\begin{cases}\left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2-1 & \text{if } n \text{ is odd,}\\ \left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2 & \text{if } n \text{ is even.}\end{cases}$$

Also, we have

$$\left(\frac{s_n[1]}{n},\ldots,\frac{s_n[1]}{n}\right)\prec(\lambda_1[1],\lambda_2[1],\ldots,\lambda_n[1]).$$

So, we have $\lambda_n[1]\le\frac{s_n[1]}{n}\le\lambda_1[1]$, i.e., if n is an odd number then

$$n\lambda_n[1]\le\left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2-1\le n\lambda_1[1],$$

if n is an even number then

$$n\lambda_n[1]\le\left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2\le n\lambda_1[1].$$

Suppose that $x\ge1$ and $(\lambda_1[x],\lambda_2[x],\ldots,\lambda_n[x])\in\mathcal{D}$. Then, from (5), we have

$$\left(3,\frac{3}{x^2},\frac{3}{x^4},\ldots,\frac{3}{x^{2n-6}},\frac{2}{x^{2n-4}},\frac{1}{x^{2n-2}}\right)\prec\left(\frac{1}{\lambda_n[x]},\frac{1}{\lambda_{n-1}[x]},\ldots,\frac{1}{\lambda_1[x]}\right). \tag{6}$$

Thus, there exists a doubly stochastic matrix $T=[t_{ij}]$ such that

$$\left(3,\frac{3}{x^2},\frac{3}{x^4},\ldots,\frac{3}{x^{2n-6}},\frac{2}{x^{2n-4}},\frac{1}{x^{2n-2}}\right)$$
$$=\left(\frac{1}{\lambda_n[x]},\frac{1}{\lambda_{n-1}[x]},\ldots,\frac{1}{\lambda_1[x]}\right)\begin{bmatrix}t_{11}&t_{12}&\ldots&t_{1n}\\t_{21}&t_{22}&\ldots&t_{2n}\\\vdots&\vdots&\ldots&\vdots\\t_{n1}&t_{n2}&\ldots&t_{nn}\end{bmatrix}.$$

So, we have

$$3=\frac{t_{11}}{\lambda_n[x]}+\frac{t_{21}}{\lambda_{n-1}[x]}+\cdots+\frac{t_{n1}}{\lambda_1[x]},$$

i.e.,

$$1=\frac{t_{11}}{3\lambda_n[x]}+\frac{t_{21}}{3\lambda_{n-1}[x]}+\cdots+\frac{t_{n1}}{3\lambda_1[x]}.$$

Since the matrix T is a doubly stochastic matrix,

$$t_{11}+t_{21}+\cdots+t_{n1}=1.$$

Lemma 3.1: Suppose that $x \geq 1$. For each $i=1,2,\ldots,n, n\geq 2$,

$$t_{n-(i-1),1} \leq \frac{3\lambda_i[x]}{n-1}.$$

Proof: Suppose that $t_{n-(i-1),1} > \frac{3\lambda_i[x]}{n-1}, i=1,2,\ldots,n$. Then

$$\begin{aligned} t_{11}+t_{21}+\cdots+t_{n1} &> \frac{3\lambda_1[x]}{n-1}+\frac{3\lambda_2[x]}{n-1}+\cdots+\frac{3\lambda_n[x]}{n-1} \\ &= \frac{3}{n-1}(\lambda_1[x]+\lambda_2[x]+\cdots+\lambda_n[x]). \end{aligned}$$

Since $x \geq 1$ and

$$\lambda_1[x]+\lambda_2[x]+\cdots+\lambda_n[x]=F_{n+1}F_n x^{2n-2}+\cdots+F_3F_2x^2+F_2F_1>n,$$

this yields a contradiction.

Therefore, $t_{n-(i-1),1} \leq \frac{3\lambda_i[x]}{n-1}, i=1,2,\ldots,n$. □

In [3], the authors found properties of the eigenvalues of $\mathcal{Q}_n[1]$ and proved the following result.

Theorem 3.2: Let $\tau = s_n[1]-(n-1)$. For $(\lambda_1[1],\lambda_2[1],\ldots,\lambda_n[1]) \in \mathcal{D}$,

$$(\tau,1,1,\ldots,1) \prec (\lambda_1[1],\lambda_2[1],\ldots,\lambda_n[1]).$$

Let $\sigma[x]=s_n[x]-\frac{n-1}{3}$. Then, we have $(\sigma[x],\frac{1}{3},\frac{1}{3},\ldots,\frac{1}{3}) \in \mathcal{D}$ and $s_n[x]=\sigma[x]+\frac{n-1}{3}=\sum_{i=1}^{n}\lambda_i[x]$. In the next theorem, we have another majorization of the eigenvalues of $\mathcal{Q}_n[x]$.

Theorem 3.3: Suppose that $x \geq 1$. For $(\lambda_1[x],\lambda_2[x],\ldots,\lambda_n[x]) \in \mathcal{D}$, we have

$$\left(\sigma[x],\frac{1}{3},\frac{1}{3},\ldots,\frac{1}{3}\right) \prec (\lambda_1[x],\lambda_2[x],\ldots,\lambda_n[x]).$$

Proof: Let $P=[p_{ij}]$ be an n by n matrix as follows:

$$P=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{12} \\ p_{21} & p_{22} & \cdots & p_{22} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{n2} \end{bmatrix},$$

where $p_{i2} = \frac{t_{n-(i-1),1}}{3\lambda_i[x]}$ and $p_{i1} = 1-(n-1)p_{i2}, i = 1,2,\dots,n$. Since T is doubly stochastic and $\lambda_i[x] > 0, p_{i2} \geq 0; i = 1,2,\dots,n$. By lemma 3.1, $p_{i1} \geq 0, i = 1,2,\dots,n$. Then

$$p_{12}+p_{22}+\cdots+p_{n2} = \frac{t_{n,1}}{3\lambda_1[x]} + \frac{t_{n-1,1}}{3\lambda_2[x]} + \cdots + \frac{t_{1,1}}{3\lambda_n[x]} = 1,$$

$p_{i1} + (n-1)p_{i2} = 1-(n-1)p_{i2} + (n-1)p_{i2} = 1$, and

$$\begin{aligned} &p_{11}+p_{21}+\cdots+p_{n1} \\ &\quad = 1-(n-1)p_{12} + 1-(n-1)p_{22} + \cdots + 1-(n-1)p_{n2} \\ &\quad = n-(n-1)(p_{12}+p_{22}+\cdots+p_{n2}) = 1. \end{aligned}$$

Thus, P is a doubly stochastic matrix. Furthermore,

$$\begin{aligned} \lambda_1[x]p_{12}+\lambda_2[x]p_{22}+\cdots+\lambda_n[x]p_{n2} &= \frac{\lambda_1[x]t_{n,1}}{3\lambda_1[x]} + \frac{\lambda_2[x]t_{n-1,1}}{3\lambda_2[x]} + \cdots + \frac{\lambda_n[x]t_{1,1}}{3\lambda_n[x]} \\ &= \frac{1}{3}(t_{n,1}+t_{n-1,1}+\cdots+t_{1,1}) = \frac{1}{3}, \end{aligned}$$

and

$$\begin{aligned} &\lambda_1[x]p_{11}+\lambda_2[x]p_{21}+\cdots+\lambda_n[x]p_{n1} \\ &\quad = \lambda_1[x](1-(n-1)p_{12})+\cdots+\lambda_n[x](1-(n-1)p_{n2}) \\ &\quad = \lambda_1[x]+\lambda_2[x]+\cdots+\lambda_n[x]-(n-1)(\lambda_1[x]p_{12}+\lambda_2[x]p_{22}+\cdots+\lambda_n[x]p_{n2}) \\ &\quad = s_n[x]-(n-1)\frac{1}{3}(t_{n,1}+t_{n-1,1}+\cdots+t_{1,1}) \\ &\quad = \sigma[x]. \end{aligned}$$

Thus, $(\sigma[x], \frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}) = (\lambda_1[x], \lambda_2[x], \dots, \lambda_n[x])P$.

Therefore,

$$\left(\sigma[x], \frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}\right) \prec (\lambda_1[x], \lambda_2[x], \dots, \lambda_n[x]). \quad \square$$

From (6), we have the following lemma.

Lemma 3.4: Suppose that $x \geq 1$. For $k = 2,3,\dots,n$,

$$\frac{1}{3(k-1)} \leq \lambda_k[x].$$

Proof: From (6), for $k \geq 2$,

$$\frac{1}{\lambda_1[x]} + \frac{1}{\lambda_2[x]} + \cdots + \frac{1}{\lambda_k[x]} \leq \frac{1}{x^{2n-2}} + \frac{2}{x^{2n-4}} + \frac{3}{x^{2n-6}} + \cdots + \frac{3}{x^{2n-2k}}.$$

Since $x \geq 1$, we have

$$\frac{1}{\lambda_1[x]} + \frac{1}{\lambda_2[x]} + \cdots + \frac{1}{\lambda_k[x]} \leq 1 + 2 + 3 + \cdots + 3 = 3(k-1).$$

Thus,

$$\frac{1}{\lambda_k[x]} \leq 3(k-1) - \left(\frac{1}{\lambda_1[x]} + \frac{1}{\lambda_2[x]} + \cdots + \frac{1}{\lambda_{k-1}[x]}\right) \leq 3(k-1).$$

Therefore, $\frac{1}{3(k-1)} \leq \lambda_k[x]$. □

In [3], the authors gave a bound for the eigenvalues of $\mathcal{Q}_n[1]$ as follows: for $k = 1, 2, \ldots, n-2$,

$$\lambda_{n-k}[1] \leq (k+1) - \frac{n-k}{3(n-1)}. \tag{7}$$

In the next theorem, we have a bound for the eigenvalues of $\mathcal{Q}_n[x]$ that is better than (7).
Theorem 3.5: Suppose that $x \geq 1$. For $k = 2, 3, \ldots, n-2$,

$$\frac{1}{3(n-k-1)} \leq \lambda_{n-k}[x] \leq \frac{1}{3}\left[k + 2 - \ln\left(\frac{n}{n-k-1}\right)\right].$$

In particular,

$$\sigma[x] \leq \lambda_1[x] \leq 3^{n-1}(n-1)!x^{n(n-1)},$$

$$\frac{1}{3(n-2)} \leq \lambda_{n-1}[x] \leq \frac{2n-3}{3(n-1)},$$

and

$$\frac{1}{3(n-1)} \leq \lambda_n[x] \leq \frac{1}{3}.$$

Proof: By theorem 3.3, we have $\sigma[x] \leq \lambda_1[x]$ and $\lambda_n[x] \leq \frac{1}{3}$. By lemma 3.4, we have $\frac{1}{3(n-1)} \leq \lambda_n[x]$. Since

$$\det \mathcal{Q}_n[x] = \det(\mathcal{R}_n[x]\mathcal{F}_n[x]^T) = x^{n(n-1)} = \lambda_1[x]\lambda_2[x]\ldots\lambda_n[x],$$

we have, by lemma 3.4,

$$\frac{1}{3^{n-1}(n-1)!} \leq \lambda_2[x] \dots \lambda_n[x].$$

Thus, $\lambda_1[x] \leq 3^{n-1}(n-1)! x^{n(n-1)}$.

By lemma 3.4, $\frac{1}{3(n-2)} \leq \lambda_{n-1}[x]$ and $\lambda_n[x] + \lambda_{n-1}[x] \leq \frac{2}{3}$. So,

$$\lambda_{n-1}[x] \leq \frac{2}{3} - \lambda_n[x] \leq \frac{2}{3} - \frac{1}{3(n-1)} = \frac{2n-3}{3(n-1)}.$$

We know that

$$\frac{1}{2} + \cdots + \frac{1}{n} \leq \int_1^n \frac{1}{x} dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n-1},$$

i.e., $\frac{1}{2} + \cdots + \frac{1}{n} \leq \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$. So, we have

$$\begin{aligned} \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{n-k} &\geq \ln n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-k-1}\right) \\ &\geq \ln n - \ln(n-k-1) - 1. \end{aligned} \tag{8}$$

Since, by (8) and

$$\lambda_{n-k}[x] \leq \frac{k+1}{3} - (\lambda_n[x] + \lambda_{n-1}[x] + \cdots + \lambda_{n-k+1}[x]),$$

we have

$$\lambda_{n-k}[x] \leq \frac{1}{3}\left[k + 2 - \ln\left(\frac{n}{n-k-1}\right)\right].$$

Therefore,

$$\frac{1}{3(n-k-1)} \leq \lambda_{n-k}[x] \leq \frac{1}{3}\left[k + 2 - \ln\left(\frac{n}{n-k-1}\right)\right]. \quad \square$$

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