LATTICE POINT SOLUTION OF THE GENERALIZED PROBLEM OF TERQUEM AND AN EXTENSION OF FIBONACCI NUMBERS

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In this paper we give a simple lattice point solution to a generalized permutation problem of Terquem and develop some elementary results for the extended Fibonacci numbers associated with the permutation problem.

The classical permutation problem of Terquem [12] has been stated by Riordan [10, p. 17, ex. 15] in the following manner. Consider combinations of n numbered things in natural (rising) order, with f(n,r) the number of r-combinations with odd elements in odd position and even elements in even positions, or, what is equivalent, with f(n,r) the number of combinations with an equal number of odd and even elements for r even and with the number of odd elements one greater than the number of even for r odd.

It is easy to show that $f(n,r) = f(n-1, r-1) + f(n-2, r)$, with $f(n, 0) = 1$, and explicitly

$$f(n, r) = \binom{\left[\frac{n+r}{2}\right]}{r}. \tag{1}$$

Moreover,

$$f(n) = \sum_{r=0}^{n} f(n, r) = f(n-1) + f(n-2) \tag{2}$$

so that f(n) is an ordinary Fibonacci number with $f(0) = 1$ and $f(1) = 2$.

A detailed discussion of Terquem's problem is given by Netto [8, pp. 84-87] and Thoralf Skolem [8, pp. 313-314] has appended notes on an extension of the problem in which the even and odd question is replaced by the more general question of what happens when one uses a modulus m to determine the position of an element in the permutation.

*Research supported by National Science Foundation Grant GP-482.

More precisely, for a modulus $m \geq 2$, Skolem's generalization may be stated as follows. From among the first n natural numbers let $f(n,r;m)$ denote the number of combinations in natural order of r of these numbers such that the j^{th} element in the combination is congruent to j modulo m. That is,

$$(3) \quad f(n,r;m) = N\left\{a_1a_2\cdots a_r\colon\ 1 \leq a_1 < a_2 < \cdots < a_r \leq n,\ a_j \equiv j \pmod{m}\right\}$$
$$= \sum_{\substack{1 \leq a_1 < a_2 < \cdots < a_r \leq n \\ a_j \equiv j \pmod{m}}} 1 \quad .$$

Consider the array in Fig. 1, where the last entry is $r + km$, with

$$k = \left[\frac{n - r}{m}\right]$$

since $r + km \leq n$ implies that the largest integral value of k cannot exceed $(n - r)/m$. This array contains those, and only those, elements from among $1, 2, \cdots, n$ which may appear in a combination. That is, the j^{th} column consists of all those elements $\leq n$ in the same congruence class (mod m) which may appear in the j^{th} position.

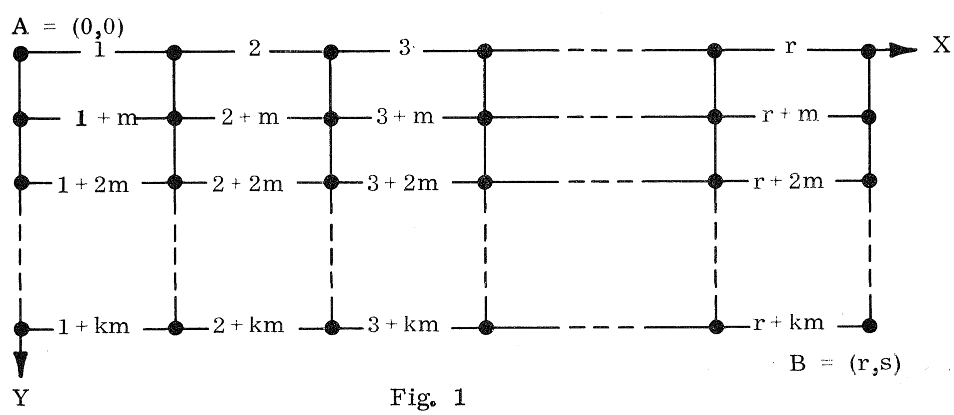


Fig. 1

From the lattice appended to the array in Fig. 1, we can systematically write out the desired combinations, and evaluate $f(n, r;m)$.

To get the desired result, let "a path from A to B" mean a path along the vertical and horizontal segments of the lattice, always moving downward or from left to right (we take the positive x-axis to the right, the positive y-axis downward, thus agreeing with the informal way of writing down the permutations). Each such path will generate a combination of the desired type, and conversely, as follows: Starting at A each horizontal step picks up an entry and vertical steps line up entries. Now, it is well known how many lattice paths there are from a = (0,0) to B = (r,s). MacMahon [7, Vol. I, p. 167] shows that this number is precisely

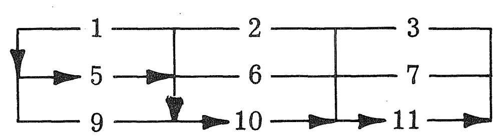
$$\binom{r + s}{r} .$$

In our case $s = [(n - r)/m]$. Thus we have at once that

$$f(n, r; m) = \binom{r + \left[\frac{n - r}{m}\right]}{r} = \binom{\left[\frac{n + (m - 1)r}{m}\right]}{r} . \tag{4}$$

as found by Skolem. Terquem's (1) follows when m = 2. To illustrate, we consider some examples.

Example 1. Let n = 12, r = 3, m = 4. Then the corresponding array is

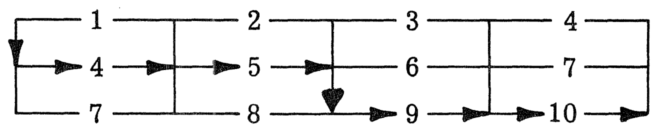


and the ten combinations are

1	2	3	1	6	7	5	6	7	9	10	11
1	2	7	1	6	11	5	6	11			
1	2	11	1	10	11	5	10	11			

and the particular combination 5, 10, 11 corresponds to the path indicated by arrows. Informally, one writes out the combinations by paths from the left column to the right column, moving horizontally and/or diagonally. The clue to a systematic count is found by superimposing the rectangular grid.

Example 2. Let n = 12, r = 4, m = 3. Then the corresponding array is



and the fifteen combinations are

1	2	3	4	1	2	9	10	4	5	6	7
1	2	3	7	1	5	6	7	4	5	6	10
1	2	3	10	1	5	6	10	4	5	9	10
1	2	6	7	1	5	9	10	4	8	9	10
1	2	6	10	1	8	9	10	7	8	9	10

and the combination 4, 5, 9, 10 corresponds to the path indicated by arrows.

It is felt that our proof shows a truism of mathematics: one may often find a simpler proof by embedding a given problem (Terquem's) in a more general setting. The lattice point enumeration we used is well known, but may not be apparent in the original problem because of its specialized form.

The extended Fibonacci numbers, in analogy to (2), are now defined by

$$f(n) = f_m(n) = \sum_{r=0}^{n} \binom{\left[\frac{n + (m-1)r}{m}\right]}{r} , \tag{5}$$

and it is not difficult to verify that they satisfy the recurrence relation

$$f_m(n) = f_m(n-1) + f_m(n-m) . \tag{6}$$

For example, with $m = 3$ we have the sequence $1, 2, 3, 4, 6, 9, 13, 19, 28, \cdots$. By well-known theorems in the theory of linear difference equations, if the distinct roots of the equation

$$t^m - t^{m-1} - 1 = 0 \tag{7}$$

are $t_1, t_2, \cdots t_m$, then there exist constants C_r such that

(8) $$f_m(n) = \sum_{r=1}^{m} C_r t_r^n \quad .$$

This generalizes the familiar formulas

$$F_n = \frac{a^n - b^n}{a - b}, \qquad L_n = a^n + b^n ,$$

for the Fibonacci-Lucas numbers. The constants C_r may be determined from the system of m linear equations in C_r:

(9) $$\sum_{r=1}^{m} C_r t_r^j = j + 1, \qquad \text{for} \qquad j = 0, 1, 2, \cdots, m - 1 \quad .$$

For example, when $m = 3$, an approximate solution of the equation (7) is given by

(10) $$\begin{cases} t_1 = 1.4655 , \\ t_2 = -0.23275 + 0.79255i , \\ t_3 = -0.23275 - 0.79255i , \end{cases}$$

where $i^2 = -1$. Relations (5) through (9) are given by Skolem [8, 313-314].

When $m = 3$ the exact solution of (7) is given by

(11) $$\begin{aligned} t_1 &= A + B + \frac{1}{3} , \\ t_2 &= \frac{1}{3} - \frac{A + B}{2} + \frac{A - B}{2}\sqrt{-3} , \\ t_3 &= \frac{1}{3} - \frac{A + B}{2} - \frac{A - B}{2}\sqrt{-3} , \end{aligned}$$

where

$$A = \frac{1}{3}\sqrt{\frac{29 + \sqrt{837}}{2}} = 1.0237 \text{ approx.}$$

$$B = \frac{1}{3}\sqrt{\frac{29 - \sqrt{837}}{2}} = 0.10854 \text{ approx.}$$

As a partial check on the values of the roots, we note the following theorem from the theory of equations. Let

$$\prod_{j=1}^{m} (t - t_j) = t^m - t^{m-1} - z \quad . \tag{12}$$

Then

$$\sum_{j=1}^{m} t_j^n = \sum_{k=0}^{\left[\frac{n}{m}\right]} A_k(n, 1 - m)\, z^k \quad , \qquad n \geq 1 \quad , \tag{13}$$

where

$$A_k(a, b) = \frac{a}{a + bk} \binom{a + bk}{k} .$$

This may be compared with the well-known [2, 3, 4, 5] expansion

$$x^a = \sum_{k=0}^{\infty} A_k(a, b) z^k \quad , \qquad \text{with } z = \frac{x - 1}{x^b} \quad , \tag{14}$$

which was actually found by Lagrange in his great memoir of 1770 (Vol. 24 of Proc. of the Berlin Academy of Sciences) and which leads at once to the general addition theorem discussed in [2, 3, 4, 5] as first noted by H. A. Rothe. See relation (20), this paper.

For the equation $t^m - t^{m-1} - z = 0$, we define the power sums of the roots t_j by

$$S(n) = \sum_{j=1}^{m} t_j^n \quad . \tag{15}$$

Since $t_j^{m-1} + z = t_j^m$, we find that

$$S(n-1) + zS(n-m) = \sum_{j=1}^{m} \left\{ t_j^{n-1} + zt_j^{n-m} \right\} = \sum_{j=1}^{m} t_j^{n-m} \left(t_j^{m-1} + z \right) = \sum_{j=1}^{m} t_j^{n-m} t_j^{m},$$

so that $S(n)$ itself also satisfies a Fibonacci-type recurrence

$$S(n) = S(n-1) + zS(n-m) . \tag{16}$$

Using the values $z = 1$, $m = 3$, the previous roots (10) yield the approximate values (by log tables): $S(1) = 1$, $S(2) = 0.9998$, $S(3) = 3.9995$, and $S(4) = 5$ very nearly. This gives a partial check on (10).

In any event, we may consider the sequence defined by (13), (15), (16) as a kind of extended Fibonacci sequence. In particular,

$$S(n) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{n}{n-(m-1)k} \binom{n-(m-1)k}{k} z^k , \qquad n \geq 1 , \tag{17}$$

satisfies (16) just as (5) satisfies (6). There are similarities and contrasts if we compare (17) and (5). We also call attention to another such result given recently by J. A. Raab [9], who found that the sequence defined by

$$x_n = \sum_{k=0}^{\left[\frac{n}{r+1}\right]} \binom{n-rk}{k} a^{n-k(r+1)} b^k \tag{18}$$

satisfies

$$x_n = ax_{n-1} + bx_{n-r-1} . \tag{19}$$

Formula (13) is substantially that given by Arthur Cayley [1]. The classical Lagrange inversion formula for series is inherent in all these formulas. One should also compare the Fibonacci-type relations here with the expansions given in [5]. For $m = 3$, (17) gives the sequence 1, 1, 4, 5, 6, 10, 15, 21, 31, ...

We also call attention to the two well-known special cases

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} 2^{n-2k} z^k = \frac{x^{n+1} - y^{n+1}}{x - y}$$

and

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \frac{n}{n-k} 2^{n-2k-1} z^k = \frac{x^n + y^n}{x + y}$$

where $x = 1 + \sqrt{z+1}$, $y = 1 - \sqrt{z+1}$. F_n and L_n occur when $z = 4$.

Relations (17) and (5) differ because the initial conditions differ. For $z = 1$, (17) satisfies precisely the same recurrence as (5). If the initial values were the same then we would have found a formula for the permutation problem not unlike (17). There are many papers (too numerous to mention) in which complicated binomial sums are found by lattice point enumerations. The convolutions in [2, 3, 4, 5] may mostly be found by such counting methods. We also note the recent papers of Greenwood [6] and Stocks [11] wherein the Fibonacci numbers occur.

The convolution addition theorem [2, 3, 4, 5] of H. A. Rothe (1793)

$$\sum_{k=0}^{n} A_k(a,b) A_{n-k}(c,b) = A_n(a+c,b) , \tag{20}$$

valid for all real or complex a, b, c (being a polynomial identity in these), has been derived several times by lattice point methods. We mention only a novel derivation by Lyness [13]. Relation (20) has been rediscovered dozens of times since 1793, and its application in probability, graph theory, analysis, and the enumeration of flexagons, etc., shows that the theorem is very useful. In fact, it is a natural source of binomial identities. We should like to raise the question here as to whether any analogous relation involving the generalized Terquem coefficients (4) exists. It seems appropriate to study the generating function defined by

$$T(x;a,b) = \sum_{n=0}^{\infty} \binom{\left[\frac{a + (b-1)n}{b}\right]}{n} x^n \tag{21}$$

for as general a and b as possible. If b is a natural number and a is an integer ≥ 0, the series terminates with that term where $n = a$, as is evident from the fact that $a + (b-1)n < bn$ for $n > a$ and the fact that $\binom{k}{n} = 0$ for $k < n$ when $n \geq 0$, provided $k \geq 0$. We also note that for arbitrary complex a and $|x| < 1$

$$T(x;a,1) = \sum_{n=0}^{\infty} \binom{a}{n} x^n = (1 + x)^a ,$$

so that in this case we do have an addition theorem:

$$T(x;a,1)T(x;c,1) = T(x;a+c,1) .$$

This, of course, corresponds to the case $b = 0$ in formula (20); the relation implies the familiar Vandermonde convolution or addition theorem.

There does not seem to be any especially simple closed sum for the series

$$C_n(a,c,b) = \sum_{k=0}^{n} \binom{\left[\frac{a + (b-1)k}{b}\right]}{k} \binom{\left[\frac{c + (b-1)(n-k)}{b}\right]}{n-k} \tag{22}$$

which occurs in

$$T(x;a,b)T(x,c,b) = \sum_{n=0}^{\infty} x^n C_n(a,c,b) \quad ,$$

for arbitrary b.

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