ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

NOTICE: PLEASE SEND ALL SOLUTIONS AND NEW PROPOSALS TO PROFESSOR RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA.

H-113 Proposed by V. E. Hoggatt, Jr., San Jose State, College, San Jose, Calif.

If

$$\sum_{n=0}^{\infty} R(n)X^n = \prod_{j=1}^{\infty}(1 + X^{F_j}) ,$$

then show

i) $$R(L_{2n} - 1) = R(L_{2n+1} - 1) = 2n \qquad n \geq 2$$

ii) $$R(L_{n+3} + 1) = 2n \qquad n \geq 2$$

(In "Representations by Complete Sequences," Oct. 1963 Fibonacci Quarterly, Theorem 3 states

$$R(L_{2n-1}) = R(L_{2n}) = 2n - 1 \quad (n \geq 1)$$

This should read $n \geq 2$.)

H-114 Proposed by William C. Lombard and V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that the sequence $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ $(n \geq 0)$ is complete.

Show that, if any L_k $(k > 2)$ is deleted, then the deleted sequence is still complete.

Show that, if L_0 or L_1 or (L_j and L_k; $k > j \geq 2$) is (are) deleted, then the deleted (doubly deleted) sequence is incomplete.

(See H-53, Vol. 3, No. 1, page 45, Fibonacci Quarterly.)

H-115 Proposed by Stephen Headley, San Jose State College, San Jose, Calif.

If

$$\sum_{n=0}^{\infty} R(n)X^n = \prod_{i=0}^{\infty} (1 + X^{L_i}) ,$$

where L_i is the i^{th} Lucas number, show $R(L_{2n}) = R(L_{2n+1}) = n + 1$.

H-116 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

If

$$\sum_{n=0}^{\infty} R(n)X^n = \prod_{j=0}^{\infty} (1 + X^{L_j}) ,$$

then for $n \geq 0$ show

i) $$R(F_{4n}) = R(F_{4n+1}) = R(F_{4n+2}) = F_{2n+1}$$

ii) $$R(F_{4n+3}) = F_{2n+2}$$

H-117 Proposed by George Ledin, Jr., San Francisco, Calif.

Prove

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} = F_{2n+6}F_{2n}$$

H-118 Proposed by George Ledin, Jr., San Francisco, Calif.

Solve the difference equation

$$C_{n+2} = F_{n+2}C_{n+1} + C_n$$

with $C_1 = a$, $C_2 = b$, and F_n is the n^{th} Fibonacci number.

SOLUTIONS

A MANY SPLENDORED THING

H-69 Proposed by M.N.S. Swamy, University of Saskatchewan, Regina, Canada

Given the polynomials $B_n(x)$ and $b_n(x)$ defined by,

$$\begin{aligned} b_n(x) &= xB_{n-1}(x) + b_{n-1}(x) \qquad (n > 0) \\ B_n(x) &= (x+1)B_{n-1}(x) + b_{n-1}(x) \qquad (n > 0) \\ b_0(x) &= B_0(x) = 1 \end{aligned}$$

It is possible to show that

$$B_n(x) = \sum_{r=0}^{n} \binom{n+r+1}{n-r} x^r \quad ,$$

and

$$b_n(x) = \sum_{r=0}^{n} \binom{n+r}{n-r} x^r \ .$$

It can also be shown that the zeros of $B_n(x)$ or $b_n(x)$ are all real, negative and distinct. The problem is whether it is possible to factorize $B_n(x)$ and $b_n(x)$. I have found that for the first few values of n, the result

$$B_n(x) = \prod_{r=1}^{n} \left[x + 4\cos^2\left(\frac{r}{n+1}\right) \cdot \frac{\pi}{2} \right]$$

holds. Does this result hold good for all n? Is it possible to find a similar result for $b_n(x)$?

Solution by John C. Sjoherg, Carlisle, Pennsylvania

Let
$$\begin{aligned} f_{2n}(x) &= b_n(x^2) \\ f_{2n+1}(x) &= xB_n(x^2) \qquad n \geq 0 \end{aligned}$$

then

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x) ,$$

with

$$f_0(x) = 1 \quad \text{and} \quad f_1(x) = x$$

and

$$f_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n-r}{r} x^{n-2r} .$$

Thus

$$f_n(2i \cos y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n-r}{r} (2i)^{n-2r}(\cos y)^{n-2r} .$$

We have by definition that

$$f_n(2i \cos y) = i^n \frac{\sin (n+1)y}{\sin y}$$

The zeros of $f_n(2i \cos y)$ are then

$$y = \frac{r\pi}{n+1} \quad \text{for} \quad r = 1, 2, 3, \cdots, n$$

and the zeros of $f_n(x)$ are then

$$x = 2i \cos \frac{r\pi}{n+1} \quad \text{for} \quad r = 1, 2, 3, \cdots, n$$

We have therefore that

$$b_n(x^2) = \prod_{r=1}^{2n} \left(x - 2i \cos \frac{r\pi}{2n+1} \right)$$

$$b_n(x) = \prod_{r=1}^{n} \left(x + 4\cos^2 \frac{r\pi}{2n+1} \right)$$

Similarly

$$B_n(x) = \prod_{r=1}^{n} \left(x + 4\cos^2 \frac{r\pi}{2n+2} \right)$$

Since

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

with

$$f_0(x) = 1 \quad \text{and} \quad f_1(x) = x ,$$

we have

$$f_n(1) = F_{n+1} .$$

Also solved by the Proposer.

NO SOLUTION

H-70 Proposed by C.A. Church, Jr., West Virginia University, Morgantown, West Va.

For $n = 2m$ show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same selection is F^2_{m+2}, and if $n = 2m + 1$, the total is $F_{m+2}F_{m+3}$.

Additional Comment by the Proposer.

A corresponding problem for circular permutations may also be posed using Kaplansky's second lemma [same reference] which leads in this case to Lucas numbers. That is, the number of ways of selecting k objects, from n arrayed on a circle, with no two consecutive is

$$\frac{n}{n-k}\binom{n-k}{k} .$$

Another solution to the problem of choosing k elements from among $1, 2, \cdots, n$ such that i and $i + 2$ do not both occur, is given by M. Abramson [Explicit Expressions for a Class of Permutation Problems, Canadian Mathematical Bulletin, 7(1964), 349]. Namely, there are

$$\sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{n - 2k + 2 + i}{k - i} \binom{k - i}{i}$$

ways. This, of course, suggests a couple of binomial identities, when his answer is compared with mine.

A VERY PRETTY RESULT

H-71 Proposed by John L. Brown, Jr., Penn. State University, State College, Pa.

Show

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} L_k = 5^n$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} F_k = 0$$

See also H-77.

Solution by the Proposer.

Let

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2} \quad ,$$

so that

$$F_k = \frac{a^k - b^k}{\sqrt{5}} \quad \text{and} \quad L_k = a^k + b^k \quad .$$

Hence

$$5^n = (2a - 1)^{2n} = \sum_{k=0}^{2n} (2a)^k (-1)^{2n-k} \binom{2n}{k}$$

and

$$5^n = (1 - 2b)^{2n} = \sum_{k=0}^{2n} (-1)^k (2b)^k \binom{2n}{k} .$$

Add to get

$$2 \cdot 5^n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^k (a^k + b^k)$$

<u>or</u>

$$\boxed{5^n = \sum_{k=0}^{2n} (-1)^k 2^{k-1} \binom{2n}{k} L_k}$$

Subtract to get

$$0 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^k (a^k - b^k)$$

<u>or</u>

$$\boxed{0 = \sum_{k=0}^{2n} (-1)^k 2^{k-1} \binom{2n}{k} F_k}$$

GENERALIZED FIBONOMIAL COEFFICIENTS

H-72 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $u_n = F_{nk}$, where F_m is the m^{th} Fibonacci number, and k is any positive integer; and let

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ m \end{bmatrix} = 1, \begin{bmatrix} m \\ n \end{bmatrix} = \frac{u_m \cdots u_1}{u_n u_{n-1} \cdots u_1 u_{m-n} u_{m-n-1} \cdots u_1}$$

then show

$$2\begin{bmatrix} m \\ n \end{bmatrix} = L_{nk}\begin{bmatrix} m-1 \\ n \end{bmatrix} + L_{(m-n)k}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}$$

This problem and many others related are thoroughly discussed in a paper, "Fibonacci Numbers and Generalized Binomial Coefficients," to appear soon in the Fibonacci Quarterly.

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CORRECTIONS

Please make the following corrections on the paper, "On a Certain Kind of Fibonacci Sums," Vol. 5, No. 1, pp. 45-58, Fibonacci Quarterly:

Page 46: In Eq. (4a), change $P_1(m,n)dx$ to $P_1(m,x)dx$

Page 49: In Corollary 1, the denominator of the second fraction should be dn instead of dn^r. Delete the first m following second $=$ sign.

Page 51: Change the first part of the last paragraph to read:

At this stage it seems clear that a study of the polynomials $P_1(m,n)$ and $P_2(m,n)$ and of the numbers $M_{1,j}$ and $M_{2,j}$ is of **basic** importance to the development of any further theory. The numbers $M_{1,j}$ and $M_{2,j}$ pose by themselves an interesting problem. The intuitive bounds...

In the last two lines, change $M_{1,j}$ to $M_{i,j}$.

Page 54: In the last line, change case to class.

Page 56: In the table title, add an asterisk to P_3, i.e., $P_3^\star(m,n)$

In the last line before Eq. (12), change written to rewritten.

Page 58: Delete the extra with in Reference 8. G. L. Jr.

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