# ON LAMÉ'S THEOREM

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We define the Fibonacci numbers $\{u_i\}$ as follows:

$$u_1 = 1,\ u_2 = 2,\ u_{n+2} = u_{n+1} + u_n \text{ for } n \geq 1 .$$

In a recent note, R. L. Duncan has shown [1] that the determination of the greatest common divisor, $(u_{n+1}, u_n)$, for any $n \geq 1$ by means of the Euclidean Algorithm always requires a number of divisions $n$ satisfying the inequality,

$$n > \frac{p_n}{\log \xi} - 5 ,$$

where $p_n$ is the number of digits in $u_n$ and

$$\xi = \frac{1 + \sqrt{5}}{2} .$$

Duncan then contrasts the classical Lamé result [2] for this case, namely

$$n < \frac{p_n}{\log \xi} + 1 ,$$

and concludes that Lamé's theorem is virtually the best possible. [Recall Lamé's theorem asserts that if a and b are positive integers, then the number of divisions, n, required to determine (a, b) by the Euclidean Algorithm satisfies the inequality,

$$n < \frac{p}{\log \xi} + 1 ,$$

where p is the number of digits in the smaller of the two integers a and b.]

Our purpose here is to show that when Lamé's bound for the number of divisions $n$ in the algorithmic determination of $(u_{n+1}, u_n)$ is written in the equivalent form

$$n \leq \left[\frac{p_n}{\log \xi}\right] + 1$$

then there exist infinitely many pairs of consecutive Fibonacci numbers $u_n$ and $u_{n+1}$ such that the determination of $(u_{n+1}, u_n)$ by the Euclidean Algorithm requires exactly

$$\left[\frac{p_n}{\log \xi}\right] + 1$$

divisions. Thus the integer 1 which appears in Lame's bound

$$\left[\frac{p_n}{\log \xi}\right] + 1 ,$$

cannot be reduced, and in this sense, Lamé's theorem cannot be improved.

From consideration of tables, we find that to determine the g. c. d. of each of the pairs, $u_5 = 8$ and $u_6 = 13$, $u_{10} = 89$ and $u_{11} = 144$, $u_{15} = 987$ and $u_{16} = 1597$, a number of divisions is required that is equal to the Lamé bound. Note that the smaller number in each pair contains exactly one less digit than the larger number; this property will also be imposed in the general analysis. It is not clear a priori that there are infinitely many such pairs for which the Lamé bound is realized. For example, the next logical pair, $u_{19} = 6765$ and $u_{20} = 10946$, requires only 19 divisions but the Lamé result gives an upper bound of 20.

THEOREM 1: There exist an infinite number of distinct positive integers $n$ such that the determination of $(u_{n+1}, u_n)$ by the Euclidean Algorithm requires exactly $n$ divisions with $n$ satisfying

$$n > \frac{p_n}{\log \xi} - \frac{1}{2} . \tag{1}$$

PROOF: It is known [1] that the algorithmic determination of $(u_{n+1}, u_n)$ requires exactly $n$ divisions; it remains to prove (1) holds for infinitely many values of $n$.

For $n \geq 1$, Binet's formula [3] states

$$u_n = \frac{\xi^{n+1} - \zeta^{n+1}}{\sqrt{5}} ,$$

where

$$\xi = \frac{1 + \sqrt{5}}{2}$$

and

$$\zeta = \frac{1 - \sqrt{5}}{2} .$$

Thus,

$$\left| u_n - \frac{\xi^{n+1}}{\sqrt{5}} \right| = \left| \frac{\zeta^{n+1}}{\sqrt{5}} \right| . \tag{2}$$

Since

$$|\zeta| < 1 , \quad \lim_{n \to \infty} \frac{\xi^{n+1}}{\sqrt{5}} = u_n .$$

Choose $\epsilon > 0$ such that

$$\frac{\log \sqrt{5}}{\log \xi} - \frac{2\epsilon}{\log \xi} > 1.5 . \tag{3}$$

[This is possible since

$$\log \sqrt{5} = 0.350$$

and

$$\log \xi = 0.208] .$$

Corresponding to this value of $\epsilon$, $\exists$ a positive integer $n_o$ such that

$$\left|\log u_n - \log \frac{\xi^{n+1}}{\sqrt{5}}\right| < \epsilon \quad \text{for} \quad n > n_o \ ,$$

or, equivalently,

$$\left|\log u_n - (n+1) \log \xi + \log \sqrt{5}\right| < \epsilon \quad \text{for} \quad n > n_o \ . \tag{4}$$

Now, $p_n$, the number of digits in $u_n$, is given by $p_n = \left[\log u_n\right] + 1$, where the square brackets denote the greatest integer contained in the bracketed quantity.

Clearly,

$$\log u_n = p_n - 1 + \theta_n \quad \text{where} \quad 0 < \theta_n < 1 \ , \tag{5}$$

and (4) becomes

$$\left|p_n - (1 - \theta_n) - (n+1) \log \xi + \log \sqrt{5}\right| < \epsilon \quad (n > n_o) \ . \tag{6}$$

Since (5) holds for arbitrary $n$, we also have

$$\log u_{n+1} = p_{n+1} - 1 + \theta_{n+1} \quad \text{with} \quad 0 < \theta_{n+1} < 1 \ , \tag{7}$$

where $p_{n+1}$ is the number of digits in $u_{n+1}$.

Subtracting (5) from (7), we find

$$\log \frac{u_{n+1}}{u_n} = (p_{n+1} - p_n) + (\theta_{n+1} - \theta_n) \ . \tag{8}$$

But it is well-known that

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \xi \ ;$$

therefore, for the previosuly chosen $\epsilon > 0$, $\exists$ a positive integer $n_o'$ such that for $n > n_o'$,

(9) $$\left|(p_{n+1} - p_n) + (\theta_{n+1} - \theta_n) - \log \xi\right| < \epsilon \; .$$

We further restrict $n$ so that

(10) $$p_{n+1} - p_n = 1$$

is satisfied; that is, $u_{n+1}$ is required to have exactly one more digit than $u_n$. Since

$$\lim_{n \to \infty} u_n = +\infty \, ,$$

it is clear that (10) is satisfied for infinitely many values of $n$. With this additional restriction on $n$, equation (9) yields

$$\theta_n > \theta_{n+1} + (1 - \log \xi) - \epsilon \; ,$$

or noting $\theta_{n+1} > 0$ ,

(11) $$\theta_n > (1 - \log \xi) - \epsilon \; .$$

From (6),

(12) $$p_n - (1 - \theta_n) - (n+1) \log \xi + \log \sqrt{5} < \epsilon \;\; (n > n_o) \; .$$

If we now choose $n > \max (n_o, n_o')$ and such that (10) is satisfied, then using (11) in (12),

(13) $$p_n - \log \xi - (n+1) \log \xi + \log \sqrt{5} - \epsilon < \epsilon \, ,$$

or

(14) $$n > \frac{p_n}{\log \xi} - 2 + \frac{\log \sqrt{5}}{\log \xi} - \frac{2 \epsilon}{\log \xi} \; .$$

Using (3), we conclude that for $n > \max (n_o, n_o')$ and satisfying (10),

(15) $$n > \frac{p_n}{\log \xi} - \frac{1}{2}$$ as asserted. q. e. d.

According to Lamé's theorem [2], the number of divisions n required to determine $(u_{n+1}, u_n)$ is bounded above (strong inequality) by

$$\frac{p_n}{\log \xi} + 1 ,$$

or equivalently

$$n \leq \left[\frac{p_n}{\log \xi}\right] + 1 . \tag{16}$$

On the other hand, (15) asserts that

$$n \geq \left[\frac{p_n}{\log \xi} - \frac{1}{2}\right] + 1 = \left[\frac{p_n}{\log \xi} + \frac{1}{2}\right] \tag{17}$$

for infinitely many values of n. Under certain circumstances, the bounds in (16) and (17) are equal. We first prove a simple lemma: (cf. [4], Theorem 6.3, p. 72):

LEMMA: Given $\alpha$ irrational, $\exists$ infinitely many integers n such that $n\alpha - [n\alpha] > \frac{1}{2}$.

PROOF: If $\exists\, n_0$ such that $n\alpha - [n\alpha] > \frac{1}{2}$ for all $n > n_0$, the proposition is proved; otherwise, for given $n_0 > 0$, $\exists$ n with $n > n_0$ such that $n\alpha = [n\alpha] + \beta$ with $0 < \beta < \frac{1}{2}$ ($\beta$ irrational).

Choose k such that

$$\frac{1}{2^{k+1}} < \beta < \frac{1}{2^k} \quad \text{or} \quad \frac{1}{2} < 2^k\beta < 1 .$$

Then, letting $N = 2^k n$, we have

$$N\alpha = 2^k[n\alpha] + 2^k\beta .$$

Since $2^k\beta < 1$, $[N\alpha] = 2^k[n\alpha]$ and $N\alpha - [N\alpha] = 2^k\beta > \frac{1}{2}$.

Thus, $\exists$ arbitrarily large integers n with $n\alpha - [n\alpha] > \frac{1}{2}$ as asserted.

The following theorem shows that Lamé's result

$$n \leq \left[\frac{p_n}{\log \xi}\right] + 1$$

is the best possible.

THEOREM 2: There exist infinitely many distinct values of $n$ such that the determination $(u_{n+1}, u_n)$ by the Euclidean Algorithm requires exactly $n$ divisions where $n$ is given by

$$n = \left[\frac{p_n}{\log \xi}\right] + 1 \tag{18}$$

PROOF: From Theorem 1, - infinitely many values of $n$ such that

$$n \geq \left[\frac{p_n}{\log \xi} + \frac{1}{2}\right]. \tag{19}$$

The proof of Theorem 1 shows that if $p_M$ is the number of digits in $u_M$ where $M = \max(n_o, n_o') + 1$, than an $n$ can be found such that $p_n$ assumes any integer value $> p_M$ and such that (19) is satisfied.

The Lemma assures us that there are infinitely many values of $p_n > p_M$ such that

$$\frac{p_n}{\log \xi} - \left[\frac{p_n}{\log \xi}\right] > \frac{1}{2} \tag{20}$$

and each of these values of $p_n$ can be combined with an appropriate value of $n$ such that (19) is satisfied. But (20) implies

$$\left[\frac{p_n}{\log \xi} + \frac{1}{2}\right] = \left[\frac{p_n}{\log \xi} + 1\right] = \left[\frac{p_n}{\log \xi}\right] + 1 .$$

Thus (19) in combination with Lamé's bound $n \leq \left[\frac{p_n}{\log \xi}\right] + 1$ shows

$$n = \left[\frac{p_n}{\log \xi}\right] + 1,$$ proving the theorem. q. e. d.

The above results have been proved using only elementary techniques. A more concise proof can be obtained using some theorems on the uniform distribution (mod 1) of sequences; this will be the subject of a forthcoming note by R. L. Duncan.

## REFERENCES


1. R. L. Duncan, Note on the Euclidean Algorithm, The Fibonacci Quarterly. Vol. 4, No. 4, pp. 367-68.
2. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill 1939, pp. 43-45.
3. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Pub. Co., 1961, p. 20.
4. I. Niven, Irrational Numbers, Carus Math. Monograph, No. 11, Math. Ass'n. of America, 1956.


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## CORRECTION

Please correct the last phrase of "A Recursive Generation on Two-Digit Integers," appearing on page 90 of the April 1965 issue of the Fibonacci Quarterly to read: "so that it takes the five odd digits to generate the set."

Edward Rayher points out that there are only nine two-digit generators. Eliminated from the published set should be "24" which obviously comes from the 21 at the end of the line preceding it in group (4), and "47" which follows 37 in the sequence of the same group.

D. R. Kapreker calls these generators "self-numbers" in his 21-page pamphlet, "The Mathematics of the New Self Numbers," personally published by him in Devlali, India in 1963. He lets the generated sequences run to infinity rather than reducing the numbers modulo 100 so that they lead to loops.

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