PYTHAGOREAN TRIANGLES AND RELATED CONCEPTS

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INTRODUCTION

The familiar Pythagorean theorem

$$a^2 + b^2 = c^2 \tag{1}$$

(a, b) = length of two sides of a right triangle
c = length of the hypotenuse

has an infinite number of integer solutions, e. g.

a	3	5	8	7	---
b	4	12	15	24	---
c	5	13	17	25	---

as Diophantus of Alexandria first demonstrated and tabulated in the third century [1]. Many of his tabulated entries, however, produce right triangles which differ only in scale, representing redundant or reducible solutions. This paper presents a method for generating only irreducible-integer ("fundamental") solutions and studies some of their common properties:

1. The hypotenuse length is always an odd number.
2. One side is always odd, one side always even.
3. The even length is always divisible by four.
4. Hypotenuse ± even side is always a perfect square.
   Hypotenuse ± odd side is always twice a perfect square.
5. Taking m and n as any distinctly odd and even numbers with no common factor (m   n), the complex quantity,

$$(m + jn)^2 = (\text{one leg}) + j\ (\text{other leg})$$

and its modulus

$$|m + jn|^2 = \text{(hypotenuse)}$$

always generate a fundamental triangle and conversely. Equivalently, the acute angles always correspond to

$$\arg.\ (m + jn)^2 = 2\tan^{-1}\left(\frac{n}{m}\right) = 2\tan^{-1} p$$

and its complement: p = twice or half of some rational number <1.

6. Any line segment of length $(2k + 1)$ or $4k$, $k = 1, 2, 3, \cdots$, constitutes a leg of at least one fundamental triangle; more in many cases.
7. The necessarily non-integer nature of solutions to $a^n + b^n = c^n$, $n > 2$, (Fermat's Last Theorem) can be proved for $n = 4k$, $k = 1, 2, 3, \cdots$.
8. In a rectangular parallelopiped of integer dimensions and integer length diagonal, two of the dimensions must be even while the third dimension and the diagonal itself must be odd.

GENERATION OF FUNDAMENTAL SOLUTIONS

One method for generating fundamental solutions rewrites (1) as

$$b = \sqrt{(c + a)(c - a)} \tag{2}$$

suggesting the special case: $(c + a) = m^2$, $(c - a) = n^2$. More generally we might set $(c + a) = r_1m^2$, $(c - a) = r_2n^2$, where neither integer r contains any repeated factors. A necessary condition for an integer solution then is $r_1 = r_2$ or equivalently, $a = r(m^2 - n^2)/2$, $b = rmn$, $c = r(m^2 + n^2)/2$.

Substitutions $r' = (r/2)$, $m' = (m + n)$ and $n' = (m - n)$ will yield equivalent expressions, except for (trivially) interchanging the roles of a and b. This equivalence helps to explain why choices of m (or m') and n (or n') as

(i) both odd numbers
(ii) both even numbers
(iii) one odd and one even

are all redundant [2]. We will choose (iii) for the most compact expressions:

$$(3) \qquad a = m^2 - n^2 \quad b = 2mn \quad c = m^2 + n^2$$

subject to further condition that m and n possess no common factors. None of the b-factors can then divide evenly into either a or c; the solution is irreducible. Appropriate choices of m and n will thus generate all fundamental solutions.

INITIAL PROPERTIES OF FUNDAMENTAL TRIANGLES

Properties 1 through 5 follow directly from Equation (3) and condition (iii). Thus,

Property 1 $\quad c = \text{hypotenuse} = m^2 + n^2 = (\text{odd}) + (\text{even}) = \text{odd}$

Property 2 $\quad a = m^2 - n^2 = (\text{odd})$

Property 2 and 3 $\quad b = 2mn = 2\ (\text{even}) = 4\ \frac{mn}{2} = 4\ (\text{integer})$

Property 4 $\quad c \pm b = (m \pm n)^2; \quad c \pm a = (2m^2; 2n^2)$

Property 5 $\quad (m + jn)^2 = (m^2 - n^2) + j2mn = a + jb$

FUNDAMENTAL TRIANGLES WITH A COMMON SIDE

In Property 6, a chosen value of

$$(4) \qquad b = 2mn = 4k((k = 1, 2, 3, \cdots)$$

will provide one or more permissible combinations of m and n, leading to as many fundamental triangles with the same even side. Resolution of $a = m^2 - n^2$ into any distinct odd factors $(m + n)(m - n)$ will likewise provide choices of

(5) $$c = \frac{(m + n)^2 + (m - n)^2}{2}$$

(6) $$b = \frac{(m + n)^2 - (m - n)^2}{2}$$

yielding fundamental triangles with the same odd side.

Regarding the number of such triangles, we may express

(7) $$mn = \frac{b}{2} = 2^x \cdot r_1^{\alpha_1} \cdot r_2^{\alpha_2} \cdot r_3^{\alpha_3} \cdots r_N^{\alpha_N} \quad ,$$

where the $r_1^{\alpha_1}$ represent distinct odd prime factors raised to an integer power. Since m and n contain no common multiple, $r_i^{\alpha_i}$ can be associated with either m or n but not both (e.g., $\alpha_i = 5$, $m \sim r_i^2$, $n \sim r_i^3$ is forbidden), giving two possible choices. The $N + 1$ factors (counting 2^x) will likewise give 2^{N+1} possible ways of expressing m and n, except that m must always identify as the larger of the two. Half of these ways, however, have simply exchanged the roles of m and n with the other half. We therefore obtain $2^{N+1}/2 = 2^N$ permissible paris of m and n, and 2^N fundamental triangles with the same even-length side.

Again,

(8) $$a = S_1^{\beta_1} \cdot S_2^{\beta_2} \cdots S_N^{\beta_N} = C \cdot D$$

where the S_i are odd, prime factors. We can similarly associate each $S_i^{\beta_i}$ with either C (odd) or D (odd) in a total of 2^N different ways. Should we specify $C = (m + n)$ as the larger and $D = (m - n)$ as the smaller, the number of distinct possibilities reduces to $(2^N)/2 = 2^{N-1}$ [3] and indicates as many fundamental triangles with the same odd side. Since a-values (odd numbers) occur twice as frequently as b-values (multiples of 4) in an ordered sequence of integers, they may quite reasonably exhibit only half the potency of b-values in generating fundamental triangles.

ORDERED TABULATIONS OF FUNDAMENTAL TRIANGLES

The a- and b-schemes in fact provide two means for ordered tabulations, viz:

b	4	8	12		16	20			60				
mn	2	4	6		8	10			30				
m	2	4	6	3	8	10	5		30	15	10	6	
n	1	1	1	2	1	1	2		1	2	3	5	
a	3	15	35	5	63	99	21		899	221	91	11	
c	5	17	37	13	65	101	29		901	229	109	61	

Table I. Illustration of b-Scheme of Tabulation

a	3	5	7	9	11	13	15			105			
m + n	3	5	7	9	11	13	15	5		105	35	21	15
m - n	1	1	1	1	1	1	1	3		1	3	5	7
b	4	12	24	40	60	84	112	8		5512	608	208	88
c	5	13	25	41	61	85	113	17		5513	617	233	137

Table II. Illustration of a-Scheme of Tabulation

These tables help to illustrate the self-evident conclusions:

1) For any specified even side, there is always a fundamental triangle whose hypotenuse and odd side differ by 2 (corresponds to $n = 1$).
2) For any specified odd side there is always a fundamental triangle whose hypotenuse and even side differ by 1 (corresponds to $m - n = 1$).

FERMAT'S THEOREM

The preceding analysis has applied the identity $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$ which might be rewritten as

(9) $$d^4 + e^4 = f^4 \quad ,$$

where $d = \sqrt{m^2 - n^2}$, $e = \sqrt{2mn}$ and $f = \sqrt{m^2 + n^2}$ corresponds to line segments as in Fig. 1.

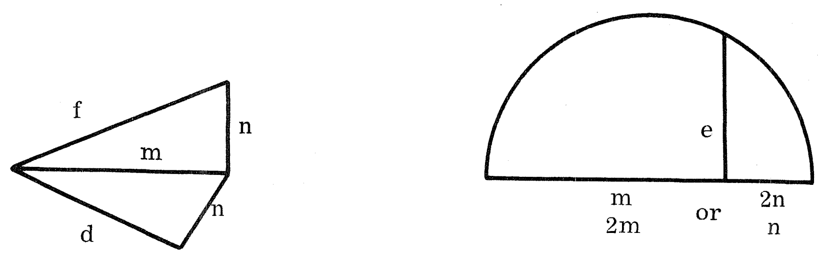

Figure 1. Graphic Constructions Expressing $d^4 + e^4 = f^4$

Thus,

(10) $$d = \sqrt{f^2 - 2n^2} = \sqrt{(f + n\sqrt{2})(f - n\sqrt{2})}$$

cannot assume integer values unless f contains a factor of $\sqrt{2}$, i.e., (9) has no integer solutions. Similarly,

(11) $$(d')^{4k} + (e')^{4k} = (f')^{4k}; \quad k = 1, 2, 3, \cdots$$

finds no integer solutions since we may set $d' = \sqrt[k]{d}$, $e' = \sqrt[k]{e}$ and $f' = \sqrt[k]{f}$.

RECTANGULAR PARALLELOPIPEDS

Some of these results apply directly to integer-sided rectangular parallelopipeds; we shall refer to Fig. 2. According to this figure,

(12) $$a^2 + b^2 = d^2 - c^2 = (d + c)(d - c)$$

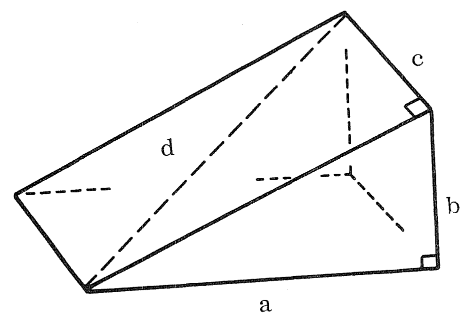

Figure 2. Diagonally Cut Half of Rectangular Parallelopiped

Suggesting that $d + c - (a^2 + b^2)/r$ and $d - c = r =$ some factor of $a^2 + b^2$, Thus,

$$(2d - r)r = (2c + r)r = a^2 + b^2 \tag{13}$$

where non-fractional values of c impose the condition

$$r_{max} \leq \sqrt{a^2 + b^2 + 1} - 1 .$$

Even values of r imply even values of a and b, while odd r demand mixed odd/even values for a and b. At least one dimension in any pair of dimensions must therefore be even; i. e., two of the three dimensions must be even. The third dimension must be odd (to prevent reducibility) while

$$d^2 = a^2 + b^2 + c^2 = (\text{even})^2 + (\text{even})^2 + (\text{odd})^2 \tag{14}$$

further requires an odd-length diagonal.

One particular scheme for generating parallelopipeds might thus begin by choosing the odd-length dimension and one of the even ones; call them and b. Evaluate $a^2 + b^2$ (now always odd) and determine the upper bound on r_{max}. Below this bound, suitable choices of r must qualify as factors of $a^2 + b^2$ and are now, likewise, always odd [4]. These choices give values of c and d via Eq. (13) and suggest the following tabulation.

Quantity	Comments	Sample Solutions						
a	One Odd, One Even	1	1		2		5	
b		2	8		9		10	
$a^2 + b^2$	Always Odd	5	65		85		125	
$r_{max} \leq$		1	7		8		10	
r	Always Odd	1	5	1	5	1	5	1
c	$= (a^2 + b^2 - r^2)/2r$	2	4	32	6	42	10	62
d	$= (a^2 + b^2 + r^2)/2r$	3	9	33	11	43	15	63

Table III. Scheme and Solutions for Fundamental Rectangular Parallelopipeds

CONCLUSION

Properties 2, 3, and 8 are the most useful since they lead directly to the generation and tabulation of fundamental solutions. The remaining properties have no such direct application but may represent areas of further study.

BIBLIOGRAPHY AND NOTES

1. G. Gamow, One, Two, Three· · · Infinity, Viking Press, New York, 1958, p. 30, presents these solutions as $a = r + \sqrt{2rs}$, $b = a + \sqrt{2rs}$, $c = r + s + \sqrt{2rs}$.
2. Choice (i) requires $r = 1$ for a fundamental solution. Choice (ii), in prime notation, requires $r' = 1/2$ and contradicts the assumption of integral r. Expressed in terms of unprimed quantities moreover it becomes indistinguishable from (i). Choice (iii) similarly requires $r = 2$. In terms of primed quantities, it also reduces to the form of (i).
3. The original 2^N ways can be grouped into pairs, indistinguishable except that one chooses C as the larger while the other chooses C as the smaller of the two factors. Specifying $C > D$ validates only one member from each pair, having the 2^N original possibilities.
4. An alternative scheme might have started with both even-length dimensions, allowing both odd and even values for r.

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