A MARKOV LIMIT PROCESS INVOLVING FIBONACCI NUMBERS

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Consider the two-state Markov chain with transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{array},$$

where $0 < a \le 1$, $0 < b \le 1$ and $0 < a + b < 2$. The branch probabilities may be displayed (Fig. 1) on a tree diagram for this chain.

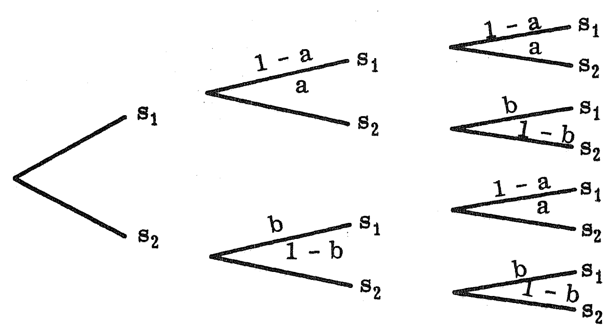

Figure 1

The above matrix P has a fixed vector $\alpha = (\alpha_1, 1 - \alpha_1)$ and limiting matrix

$$A = \lim_{n\to\infty} P^n = \begin{pmatrix} \alpha_1 & 1-\alpha_1 \\ \alpha_1 & 1-\alpha_1 \end{pmatrix}$$

for $0 < \alpha_1 < 1$. The entries α_1 and $(1 - \alpha_1)$ may be interpreted as the limiting proportion of times that the process is in state s_1 and s_2, respectively, as the number of steps, n, increases indefinitely.

For example, the Markov chain transition matrix

$$P = \begin{array}{cc} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{pmatrix} \end{array}$$

has fixed vector

$$\alpha = \left(\frac{3}{11} , \frac{8}{11} \right)$$

and the process would be expected to be in states s_1 and s_2 in a ratio of 3:8 as the number of steps increases indefinitely.

For the special case

$$P = \begin{array}{cc} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0 & 1 \\ b & 1 - b \end{pmatrix} \end{array}$$

$(0 \le b \le 1)$, the fixed vector α is

$$\alpha = \left(\frac{b}{1 + b} , \frac{1}{1 + b} \right) ,$$

using the indicated branch probabilities from the matrix P. However, we delete the branch probabilities from the tree and display only the "bare" branches of the tree (Fig. 2).

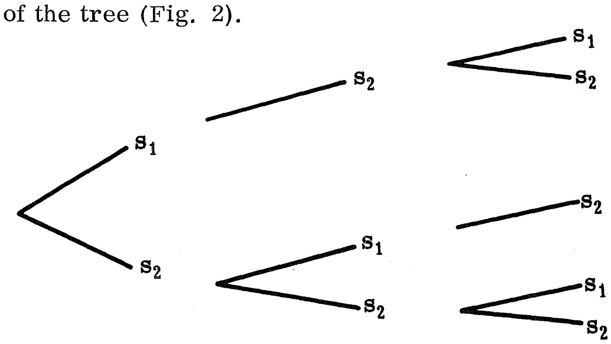

Figure 2

Instead of reading "out" the tree (i.e., left to right), we read "across" the tree (i.e., from top to bottom), so that in the first step, s_1 appears once and s_2 appears twice; in the second step s_1 appears twice and s_2 appears three times, and so on.

Denote by $N(1,n)$ the number of times that s_1 appears in the n^{th} stage, with $N(2,n)$ defined similarly. $(n = 0, 1, 2, 3, \cdots)$. It is apparent that $N(1,n) = u_n$, where u_n is the $(n+1)^{st}$ Fibonacci number in the sequence $u_0 = 1$, $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $u_4 = 5$, etc. Similarly, the total number of entries in the n^{th} stage is given by $N(1,n) + N(2,n) = u_{n+2}$, and also read $N(2,n) = u_{n+2} - N(1,n) = u_{n+2} - u_n = u_{n+1}$. Thus, continuing to read "across" the tree, the proportion of times that s_2 appears in the n^{th} stage is

$$\frac{N(2,n)}{N(1,n) + N(2,n)} = \frac{u_{n+1}}{u_{n+2}} ,$$

and this proportion has a well-known limiting value

$$\alpha = \frac{\sqrt{5} - 1}{2} = \lim_{n \to \infty} \frac{u_{n+1}}{u_{n+2}} .$$

Thus, consider the case

$$P = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{\sqrt{5} - 1}{2} & \frac{3 - \sqrt{5}}{2} \end{pmatrix} \end{array} .$$

The fixed vector is

$$\alpha = \left(\frac{3 - \sqrt{5}}{2} , \frac{\sqrt{5} - 1}{2} \right) = (.382\cdots, .618\cdots) ,$$

and so the process would be expected to be in state s_2 approximately 61.8% of the time, as n increases indefinitely, using the indicated branch probabilities. On the other hand, using only the corresponding "bare" tree, the limiting proportion of times that s_2 will appear in the n^{th} "column" (stage) will be the same proportion, as the number of steps increases indefinitely.

REFERENCES

1. J. G. Kemeny and J. L. Snell, Finite Markov Chains, Princeton, D. van Nostrand Co., Inc., 1960.

2. N. N. Vorob'ev, Fibonacci Numbers, New York, Blaisdell Publishing Company, 1961.

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CORRECTIONS

Please make the following corrections on the paper, "On Summation Formulas and Identities for Fibonacci Numbers," Vol. 5, No. 1, pp. 1-43, Fibonacci Quarterly:

Page 22: In Eq. (4.3), change $b = 0$ to read $b \neq 0$.
In the reference on the last line, add parentheses around 12.

Page 36: In the first line of Eq. (5.23), change c_{j-1} to read c_{j-i}.

Page 38: In the second line, insert brackets around the reference.
In the first line following Eq. (6.3), change i = 1) to i = 1,2

Page 39: In the first line following Eq. (6.5), add brackets around reference.

Page 40: In Eq. (6.15), change $P_1(m,n)$ to read $P_1(m,-n)$. D. Z.

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NOTICE

George Ledin, Jr. has been appointed by The Fibonacci Association to collect and classify all existing Fibonacci Identities, Lucas Identities, and Hybrid Identities. We request that readers send copies of their private lists (with possible reference sources) to

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for inclusion in the planned booklet.

Verner E. Hoggatt, Jr.,
Director

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