SOME RABBIT PRODUCTION RESULTS INVOLVING FIBONACCI NUMBERS

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Let us consider a pair of rabbits born in the 0-th month which produce B_1 offspring pairs when they are one month old, B_2 offspring pairs when they are two months old and so on. The sequence of numbers

$$B_1, B_2, B_3, \ldots B_n, \ldots$$

is called the birth sequence, and let its generating function be

$$B(x) = \sum_{n=0}^{\infty} B_n x^n ,$$

where $B_0 = 0$.

Suppose each pair of offspring also produces B_n offspring pairs when it is n months old. Let the number of new arrivals at the n-th month be Rn, and let

$$R(x) = \sum_{n=0}^{\infty} R_n x^n$$

where $R_0 = 1$. Let the total number of rabbits alive at the end of the n-th month be T_n, and let

$$T(x) = \sum_{n=0}^{\infty} T_n x^n$$

where $T_0 = 1$. We will assume that there are no deaths.

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It has been shown (see [1], [2]) that

$$R(x) = \frac{1}{1-B(x)}$$

and

$$T(x) = \frac{1}{(1-x)(1-B(x))} .$$

The purpose of this paper is to show some particular cases in which there are interesting relationships between $B(x)$, $R(x)$, and $T(x)$.

When

$$B(x) = \sum_{n=0}^{\infty} x^{n+2} ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1} x^n$$

When

$$B(x) = \sum_{n=2}^{\infty} (2n-1)x^n ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^2 x^n .$$

When

$$B(x) = \sum_{n=2}^{\infty} (6C_{n-1} + 1)x^n ,$$

where the C_n are terms of the Pell sequence defined by $C_0 = 0$, $C_1 = 1$, $C_{n+2} = 2C_{n+1} + C_n$, then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^3 x^n .$$

It is conjectured that when

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^p x^n ,$$

the corresponding $B(x)$ will have $B_n \geq 0$ for all n. This has been demonstrated for $p \leq 7$.

Hoggatt showed in [1], section 4, that when

$$B(x) = \frac{F_{k+1}x-(-1)^k x^2}{1-F_{k-1}x} , \tag{1}$$

then

$$R(x) = \sum_{n=0}^{\infty} F_{kn+1}x^n ,$$

and similarly when

$$B(x) = \frac{F_{k-1}x-(-1)^k x^2}{1-F_{k+1}x} \tag{2}$$

then

$$R(x) = \sum_{n=0}^{\infty} F_{kn-1}x^n .$$

By merely changing the sign of the second term of the numerator of equations (1) and (2), we obtain the following results, which depend on the parity of k. When

$$B(x) = \frac{F_{k+1}x+(-1)^k x^2}{1-F_{k-1}x} \quad ,$$

then for k odd we have

$$R(x) = 1 + \sum_{n=0}^{\infty} \left[U_{n+1}(L_k/2) - F_{k-1}U_n L_k/2)\right] x^{n+1} \quad , \tag{3}$$

where the $U_n(x)$ are Chebyshev polynomials of the second kind defined by $U_0(x) = 0$, $U_1(x) = 1$, $U_{n+2}(x) = 2xU_{n+1}(x) + U_n(x)$.

For k even we have

$$R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(L_k) - F_{k-1}f_n(L_k)\right] x^n \quad , \tag{4}$$

where the $f_n(x)$ are the Fibonacci polynomials defined by $f_0 = 0$, $f_1 = 1$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$. Similarly when

$$B(x) = \frac{F_{k-1}x+(-1)^k x^2}{1-F_{k+1}x}$$

then for k odd we get

$$R(x) = 1 + \sum_{n=0}^{\infty} \left[U_{n+1}(L_k/2) - F_{k+1}U_n(L_k/2)\right] x^{n+1} \tag{5}$$

while for k even we find

$$(6) \qquad R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(L_k) - F_{k+1} f_n(L_k)\right] x^n \quad ,$$

where $U_n(x)$ and $f_n(x)$ are defined above.

Two other possibilities occur when L_k is substituted for F_k in equations (1) and (2). When

$$B(x) = \frac{L_{k+1}x + (-1)^k x^2}{1 - L_{k-1}x}$$

then for k odd

$$(7) \qquad R(x) = 1 + \sum_{n=0}^{\infty} \left[U_{n+1}(5/2\ F_k) - L_{k-1} U_n(5/2\ F_k)\right] x^{n+1} \quad .$$

For k even,

$$(8) \qquad R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(5F_k) - L_{k-1} f_n(5F_k)\right] x^n \quad .$$

Similarly, when

$$B(x) = \frac{L_{k-1}x + (-1)^k x^2}{1 - L_{k+1}x}$$

then for k odd,

$$R(x) = 1 + \sum_{n=0}^{\infty} \left[U_{n+1}(5/2\ F_k) - L_{k+1} U_n(5/2\ F_k)\right] x^{n+1}$$

and for k even,

$$R(x) = \sum_{n=0}^{\infty} \left[f_{n+1}(5F_k) - L_{k+1} f_n(5F_k)\right] x^n \ .$$

Note that equations (7) through (10) are the Lucas duals to equations (3) through (6).

REFERENCES

1. V. E. Hoggatt, Jr., Generalized Rabbits for Generalized Fibonacci Numbers, to appear, The Fibonacci Quarterly.
2. V. E. Hoggatt, Jr. and D. A. Lind, The Dying Rabbit Problem, to appear The Fibonacci Quarterly.

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