SIMULTANEOUS PRIME AND COMPOSITE MEMBERS IN TWO ARITHMETIC PROGRESSIONS

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Theorem. Any one of two non-identical infinite reduced arithmetic progressions has an infinitude of prime members the corresponding members of which in the other arithmetic progression are composite.

Proof. Be

$$(1) \quad ax + b, \quad (a,b) = 1, \quad x = 1,2,3,\cdots$$
$$(2) \quad cx + d, \quad (c,d) = 1, \quad x = 1,2,3,\cdots \qquad a \neq c \text{ or } b \neq d$$

two non-identical infinite arithmetic progressions. We may suppose, without loss of generality, $a \geq 1$, $c \geq 1$, since if, say, $a < 1$, we can consider the progression $-ax - b$, the members of which have the same absolute values as the corresponding members of (1). Suppose, contrary to the assertion of the theorem, that one of the progressions, say (1), has only a finite number of prime members the corresponding members of which in the other progression are composite. Thus, there is a positive integer N such that $N > |d|$ in case $a = c$, and

$$N > \max\left(\frac{d - b}{a - c}, |d|\right)$$

in case $a \neq c$, and such that, for any positive integer $x > N$, $cx + d$ is a prime if $ax + b$ is a prime. By Dirichlet, (1) has an infinitude of prime members. Hence, there is a positive integer $x_0 > N$ such that $ax_0 + b$ is a prime, whence, by the assumption, also $cx_0 + d$ is a prime. If $a = c$, then $b \neq d$, and $ax + b \neq cx + d$ for any x. If $a \neq c$, then $ax + b - cx + d$ only for $x = (d - b)/a - c)$. Since $x_0 > N > (d - b)/(a - c)$ we have $ax_0 + b \neq cx_0 + d$. Thus the arithmetic progression $a(cx_0 + d)x + (ax_0 + b)$, $x = 1,2,3,\cdots$, is reduced. Hence, by Dirichlet, there is a positive integer x_1 such that $a(cx_0 + d)x_1 + (ax_0 + b)$ is a prime. Now put $x_2 = (cx_0 + d)x_1 + x_0$. Since $c \geq 1$, $x_0 > N > |d|$, $x_1 \geq 1$, we have $x_2 > x_0 > N$. Thus $ax_2 + b = a(cx_0 + d)x_1 + (ax_0 + b)$ is a prime with $x_2 > N$, while $cx_2 + d = c(cx_0 + d)x_1 + (cx_0 + d) = (cx_0 + d)(cx_1 + 1)$ is evidently composite with $x_2 > N$, since both $cx_0 + d$, being a prime, and $cx_1 + 1$, with $c \geq 1$, $x_1 \geq 1$, are integers > 1. The contradiction to the assumption thus obtained proves the theorem.

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