IDENTITIES FOR PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

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The Fibonacci numbers F_n and Lucas numbers L_n may be defined by

(1) $$F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

where n is any integer, $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \frac{1}{2}(1 - \sqrt{5})$, so that

(2) $$\alpha - \beta = \sqrt{5} \quad \text{and} \quad \alpha\beta = -1 .$$

Recently Brother Alfred Brousseau asked for generalizations of the identity

$$F_{2k}F_{2\ell} = F^2_{k+\ell} - F^2_{k-\ell}$$

proved by I. D. Ruggles in [1], and as a result V. E. Hoggatt, D. Lind, C. R. Wall [2], and Sheryl B. Tadlock [3] between them gave seven further identities. In this note we point out that these eight identities belong to the family of sixteen identities given in Theorem 1 below. Furthermore, we show that this theorem can be proved by a very simple method which can be used to generate identities for arbitrary products of Fibonacci and Lucas numbers.

Theorem 1. If i, j, s, and t are integers such that $i + j = 2s$ and $i - j = 2t$, then

(3) $$(5F_iF_j \text{ or } L_iL_j) = (5F_s^2 \text{ or } L_s^2) \pm (5F_t^2 \text{ or } L_t^2) \pm (0 \text{ or } 4(-1)^t) .$$

where either term may be chosen at will in the first three brackets; in addition i, j may be chosen as being either both even or both odd. The choice of term in the last bracket and the sign preceding it depend on the combination of the previously mentioned four choices, but the choice of the first $\pm$ sign depends only on the parity of i and on the term chosen from the first bracket. If we fix the two choices to be made on the left side of the identity, then the four identities obtained by varying the choices made on the right side are deducible from each other by application of the well-known identity

$$5F_n^2 = L_n^2 - 4(-1)^n .$$

This fact was used by Sheryl B. Tadlock in [3].

To obtain further identities such as those of Theorem 1, we consider an arbitrary product of Fibonacci and Lucas numbers. In other words we let $m, n, i_0, i_1, \cdots, j_1, j_2, \cdots$ be any integers with $m, n \geq 0$ and put

$$P = 5^m F_{i_1} F_{i_2} \cdots F_{i_{2m}} L_{j_1} L_{j_2} \cdots L_{j_n} , \tag{4}$$

$$Q = 5^m F_{i_0} F_{i_1} \cdots F_{i_{2m}} L_{j_1} L_{j_2} \cdots L_{j_n} . \tag{5}$$

Notice that P contains an even and Q an odd number of Fibonacci numbers F_i.

First we discuss P. By (2) we have $5^m = (\alpha - \beta)^{2m}$, so substituting from (1) in (4) and expanding we see that P is symmetric in α, β and is therefore a sum of 2^{2m+n-1} terms of the form $\pm(\alpha^p \beta^q + \alpha^q \beta^p)$. But by (1), (2) we have

$$\alpha^p \beta^q + \alpha^q \beta^p = (\alpha\beta)^q (\alpha^{p-q} + \beta^{p-q}) = (-1)^q L_{p-q} . \tag{6}$$

Hence P can be expressed as the difference of two sums of Lucas numbers.

Now suppose that the sum of the subscripts occurring in (4) is even, so that we have

$$i_1 + i_2 + \cdots + i_{2m} + j_1 + j_2 + \cdots + j_n = 2s \tag{7}$$

for some integer s. Then for each of the terms $\pm(\alpha^p\beta^q + \alpha^q\beta^p)$ in the expansion of (4) we have $p + q = 2s$, and so $p - q$ is also even. Putting $p - q = 2t$ in (6) and noting that

$$\begin{aligned} \alpha^{p-q} + \beta^{p-q} = \alpha^{2t} + \beta^{2t} &= (\alpha^t - \beta^t)^2 + 2(\alpha\beta)^t = 5F_t^2 + 2(-1)^t \\ &= (\alpha^t + \beta^t)^2 - 2(\alpha\beta)^t = L_t^2 - 2(-1)^t , \end{aligned}$$

we see that our general term is of the form

$$\begin{aligned}\pm(\alpha^p\beta^q + \alpha^q\beta^p) &= \pm(-1)^q L_{2t} = \pm(-1)^q[5F_t^2 + 2(-1)^t] \\ &= \pm(-1)^q[L_t^2 - 2(-1)^t].\end{aligned}$$

Thus we have shown that the product P given by (4) can be expressed, in a large number of ways, as a sum of terms $\pm(5F_t^2$ or $L_t^2)$ and terms ± 2. As an example we give the identity

$$\begin{aligned}5F_{2i}F_{2j}L_{2k}L_{2h} &= 5F^2_{i+j+k+h} + L^2_{i+j+k-h} + 5F^2_{i+j-k+h} + L^2_{i+j-k-h} \\ &\quad -5F^2_{i-j+k+h} - 5F^2_{i-j+k-h} - L^2_{i-j-k+h} - L^2_{i-j-k-h}\,.\end{aligned}$$

The right-hand side of this identity is of course only one of the 2^8 possible expressions of this form, though many of these would involve a further term 4C, where C is an integer in the range $-4 \le C \le 4$.

Finally we discuss the product Q given by (5). Substituting from (1) and expanding we see that Q is a sum of 2^{2m+n} terms of the form

$$\pm(\alpha^p\beta^q - \alpha^q\beta^p)/(\alpha - \beta) = \pm(-1)^q F_{p-q}\ ,$$

so that Q can be expressed in terms of Fibonacci numbers F_u. There is no immediate analogue for Q to the results obtained by (7) for P. However, if each of $i_0, i_1, \cdots, j_1, j_2, \cdots$ is divisible by 3, then in our general term we have $p - q = 3r$ and

$$F_{p-q} = (\alpha^{3r} - \beta^{3r})/(\alpha - \beta) = [5F_r^3 + 3(-1)^r F_r].$$

In this way one can obtain Q as a sum of terms $\pm[5F_r^3 + 3(-1)^r F_r]$, for example (after some re-arranging of terms) we have

$$\begin{aligned}F_{3i}L_{3j}L_{3k} &= 5[F^3_{i+j+k} + (-1)^k F^3_{i+j-k} + (-1)^j F^3_{i-j+k} - (-1)^i F^3_{-i+j+k}] + \\ &\quad + 3(-1)^{i+j+k}[F_{i+j+k} + (-1)^k F_{i+j-k} + (-1)^j F_{i-j+k} - (-1)^i F_{-i+j+k}].\end{aligned}$$

REFERENCES

1. I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," The Fibonacci Quarterly, Vol. 1, No. 2 (1963), p. 77.
2. Solution to Problem B-22, "Lucas Analogues," The Fibonacci Quarterly, Vol. 2, No. 1 (1964), p. 78.
3. Sheryl B. Tadlock, "Products of Odds," The Fibonacci Quarterly, Vol. 3, No. 1 (1965), pp. 54-56.

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All subscription correspondence should be addressed to Brother Alfred Brousseau, St. Mary's College, Calif. All checks ($1.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to V. E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscript sent to the editors.

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To the Editor:

The lemma proven by M. Bicknell and V. E. Hoggatt,Jr. in 'Fibonacci Matrices and Lambda Functions',The Fibonacci Quarterly, Vol. 1 (April 1963),page 49 was essentially established in my note 'Theorem on Determinants', Mathematics Magazine Vol. 34 (September 1961), page 328. Namely, 'If the difference of each pair of corresponding elements of any two columns (rows) of a determinant are equal, then any quantity may be added to each element of the determinant without changing its value.'

Charles W. Trigg