FIBONACCI FUNCTIONS

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1. INTRODUCTION

There is a sequence of continuous functions of one variable having many of the properties of the Fibonacci sequence of numbers, with some intriguing variations. Derivatives and integrals of these functions are easily found, and lead to more relations involving Fibonacci numbers. Other topics of calculus can undoubtedly be applied to these functions with very useful and interesting results.

Let $a_0, a_1, a_2, \cdots$ be a sequence such that $a_{m+1} = a_m + a_{m-1}$. Then the power series

$$y = \sum_{i=0}^{\infty} \frac{a_i x^i}{i!}$$

satisfies the differential equation

$$y'' - y' - y = 0 \tag{1}$$

whose solution is $c_1 e^{\alpha x} + C_2 e^{\beta x}$, where α and β are the roots of $u^2 - u - 1 = 0$,

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2} .$$

If the sequence $\{a_m\}$ is the Fibonacci sequence $\{F_m\}$, then $a_0 = 0$ and $a_1 = 1$, so that we get the boundary conditions $x = 0$, $y = 0$, $y' = 1$. This yields (see [1])

$$y = \frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}} \quad \text{and} \quad F_m = \frac{\alpha^m - \beta^m}{\sqrt{5}} \tag{2}$$

On the other hand, if the sequence $\{a_m\}$ is the Lucas sequence $\{L_m\}$, then $a_0 = 2$ and $a_1 = 1$, so that we get the boundary conditions $x = 0$, $y = 2$, $y' = 1$, yielding

$$y = e^{\alpha x} + e^{\beta x} \quad \text{and} \quad L_m = \alpha^m + \beta^m \tag{3}$$

The writing of (1) in the form $y'' = y' + y$ is very suggestive: the sum of the function and its first derivative is the second derivative. And generally, if y is any solution of (1), we see that

$$y^{(m+1)} = y^{(m)} + y^{(m-1)} \tag{1a}$$

This suggests that we use the notation

$$f_0(x) = \frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}}, \quad f_1(x) = f_0'(x), \quad f_2(x) = f_0''(x), \quad f_3(x) = f_0^{(3)}(x),$$

and so forth. Thus

$$f_m(x) = f_0^{(m)}(x) = \frac{\alpha^m e^{\alpha x} - \beta^m e^{\beta x}}{\sqrt{5}} \tag{4}$$

giving us the sequence of functions $\{f_m(x)\}$ with the property that

$$f_{m+1}(x) = f_m(x) + f_{m-1}(x) \tag{5}$$

We shall refer to these functions as Fibonacci functions.

Likewise if $l_0(x) = e^{\alpha x} + e^{\beta x}$, $l_1(x) = l_0'(x)$, $l_2(x) = l_0''(x)$, etc., we have

$$l_m(x) = l_0^{(m)}(x) = \alpha^m e^{\alpha x} + \beta^m e^{\beta x} \tag{6}$$

$$l_{m+1}(x) = l_m(x) + l_{m-1}(x) \tag{7}$$

and these functions will be called Lucas functions here.

Evidently, $F_0 = f_0(0) = 0$, $F_1 = f_1(0) = 1$, $F_2 = f_2(0) = 1$, $F_3 = f_3(0) = 2, \cdots$,

$$F_m = f_m(0) = \frac{\alpha^m - \beta^m}{\sqrt{5}} \tag{8}$$

with similar results for Lucas numbers.

Let us define

$$f_{-k}(x) = \frac{\alpha^{-k}e^{\alpha x} - \beta^{-k}e^{\beta x}}{\sqrt{5}}, \qquad l_{-k}(x) = \alpha^{-k}e^{\alpha x} + \beta^{-k}e^{\beta x} \tag{9}$$

With the understanding that $f_0^{(-k)}(x)$ is the k^{th} antiderivative of $f_0(x)$, and similarly for $l_0^{(-k)}(k)$, we can easily verify that all the preceding results (2) through (8) hold for m a negative integer.

2. GRAPHS

Elementary notions of calculus regarding intercepts, slope, symmetry, extent, critical points, points of inflection, etc., may be used in plotting the graphs of these functions. Figure 1 shows the graphs of some of the Fibonacci functions $f_m(x)$.

Note first of all that the y-intercept of the curve $y = f_m(x)$ is F_m.

Observe also that the functions with even subscripts are monotonic increasing, and extend from $-\infty$ to $+\infty$ both horizontally and vertically. The functions with odd subscripts, however, are never negative (since $\beta < 0$), and each has one relative minimum.

In fact, $f_{2k-1}(x)$, where k is any integer, has its relative minimum at the zero of $f_{2k}(x)$, which is also the x at which $f_{2k-2}(x)$ has its point of inflection.

Let us therefore call these points x_{2k}. That is, x_{2k} is such that

$$f_{2k}(x_{2k}) = 0 \tag{10}$$

Let the minima of $f_{2k-1}(x)$ be called y_{2k}. Thus

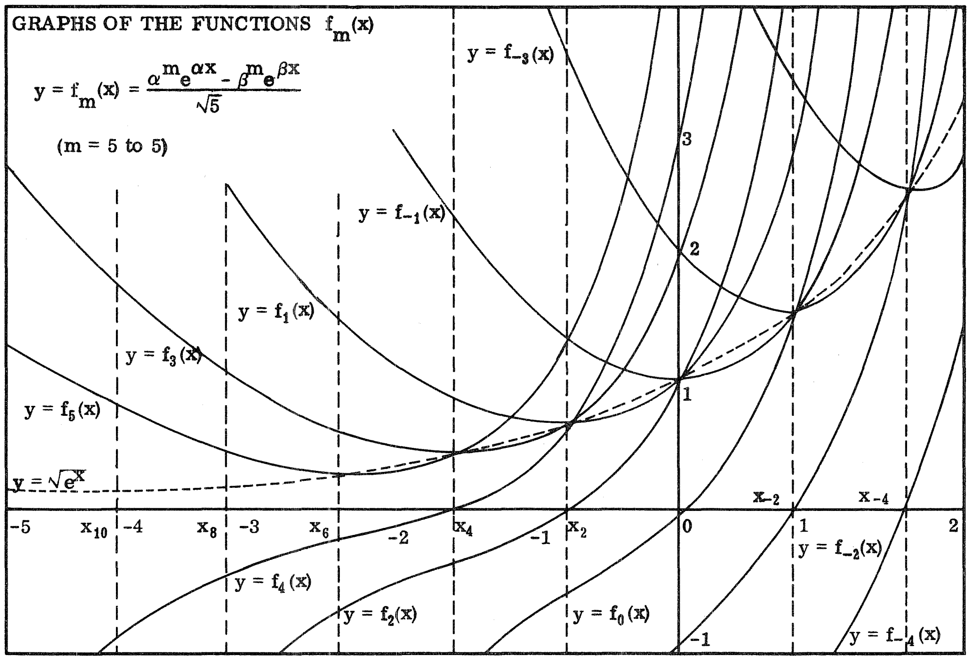

Fig. 1

(11) $$y_{2k} = f_{2k-1}(x_{2k})$$

Some manipulation and calculation result in

(12) $$x_{2k} = \frac{4k}{\sqrt{5}} \log \frac{1}{\alpha} \approx -0.86k$$

(13) $$y_{2k} = \left(\frac{1}{\alpha}\right)^{\frac{2k}{\sqrt{5}}} = e^{x_k} \approx (0.65)^k, \quad \text{where} \quad x_k = \frac{1}{2} x_{2k}$$

Thus the minima of $f_{2k-1}(x)$ occur at points evenly spaced along the negative x-axis and have values in geometric progression, which approach 0 as $x \to -\infty$.

Because

$$y_{2k} = e^{\frac{1}{2}x_{2k}}$$

at these minimum points, they lie on the graph of $y = \sqrt{e^x}$ (see the dotted-line curve in Fig. 1).

Since

$$f_{2k+1}(x_{2k}) = f_{2k}(x_{2k}) + f_{2k-1}(x_{2k}) = 0 + y_{2k} = y_{2k}$$

and since

$$f_{2k+2}(x_{2k}) = f_{2k+1}(x_{2k}) + f_{2k}(x_{2k}) = y_{2k} + 0 = y_{2k}$$

we see that the graphs of

$$f_{2k-1}(x), \quad f_{2k+1}(x), \quad \text{and} \quad f_{2k+2}(x)$$

all intersect at (x_{2k}, y_{2k}).

Likewise

$$f_{2k+3}(x_{2k}) = f_{2k+2}(x_{2k}) + f_{2k+1}(x_{2k}) = 2y_{2k} ,$$

etc.; and induction leads to

$$(14) \qquad f_{2k+j}(x_{2k}) = F_j y_{2k} = F_j e^{x_k}, \quad \text{or} \quad f_m(x_{2k}) = F_{m-2k} y_{2k} = F_{m-2k} e^{x_k}$$

which is a specialization of the more general relation to be derived in the next section.

3. AN IMPORTANT IDENTITY

That the identity

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$$

for Fibonacci numbers has a counterpart for the Fibonacci functions can be investigated by substituting into its right side:

$f_{m-1}(x)f_n(x) + f_m(x)f_n(x) =$

$$\frac{\alpha^{m-1}e^{\alpha x} - \beta^{m-1}e^{\beta x}}{\sqrt{5}} \cdot \frac{\alpha^n e^{\alpha x} - \beta^n e^{\beta x}}{\sqrt{5}} + \frac{\alpha^m e^{\alpha x} - \beta^m e^{\beta x}}{\sqrt{5}} \cdot \frac{\alpha^{n+1}e^{\alpha x} - \beta^{n+1}e^{\beta x}}{\sqrt{5}}$$

Multiplying and simplifying using $\alpha\beta = -1$, the terms in $e^{\alpha x + \beta x}$ vanish, giving us

$$\frac{\alpha^{m+n-1}(1 + \alpha^2)e^{2\alpha x} + \beta^{m+n-1}(1 + \beta^2)e^{2\beta x}}{5}$$

whence

$$1 + \alpha^2 = \alpha\sqrt{5}, \; 1 + \beta^2 = -\beta\sqrt{5}$$

lead to

$$f_{m-1}(x)f_n(x) + f_m(x)f_{n+1}(x) = f_{m+n}(2x)$$

We see then that the formula is the same except for the important change in the argument. We generalize this by repeating almost exactly the same steps, and obtain

$$f_{m+n}(u + v) = f_{m-1}(u)f_n(v) + f_m(u)f_{n+1}(v) \tag{15}$$

4. APPLICATION OF (15) TO GRAPHS

Using the identity (15) with $m = 2k$, $n = 0$, $u = x_{2k}$, and $v = t$, we obtain

$$f_{2k}(x_{2k} + t) = f_{2k-1}(x_{2k})f_0(t) + f_{2k}(x_{2k})\, f_1(t) = y_{2k}f_0(t) + 0 \cdot f_1(t)$$

$$f_{2k}(x_{2k} + t) = e^{x_k}f_0(t) \tag{16}$$

Similarly

$$f_{2k+1}(x_{2k} + t) = e^{x_k} f_1(t) \tag{17}$$

Hence each of the graphs can be obtained from the graph of either $y = f_0(x)$ or $y = f_1(x)$, according to whether m is even or odd, by expanding it by the factor e^{x_k} and translating it $-x_{2k}$ units to the left.

Since $f_0(x)$ and $f_1(x)$ in turn can be written as

$$f_0(x) = \frac{2e^{\frac{x}{2}}}{\sqrt{5}} \sinh \frac{\sqrt{5}}{2} x, \quad f_1(x) = \frac{2e^{\frac{x}{2}}}{\sqrt{5}} \cosh\left(\frac{\sqrt{5}}{2} x + \cosh^{-1}\left(\frac{\sqrt{5}}{2}\right)\right) \tag{18}$$

all of the graphs are distortions of hyperbolic sine or cosine curves through multiplication by $\sqrt{e^x}$.

5. INTEGRALS

From the definition of $f_m(x)$, the antiderivative of $f_m(x)$ is $f_{m-1}(x)$. This leads to a wealth of problems involving Fibonacci numbers, two of which follow. (See Fig. 2.)

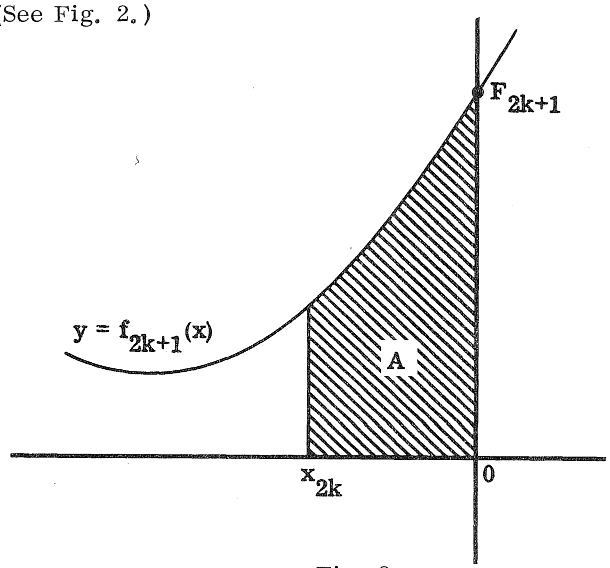

Fig. 2

Let A = the area under the curve $f_{2k+1}(x)$ between $x = x_{2k}$ and $x = 0$, and above the x-axis. Then

$$A = \int_{x_{2k}}^{0} f_{2k+1}(x)dx = f_{2k}(x)\Big]_{x_{2k}}^{0} = f_{2k}(0) - f_{2k}(x_{2k}) = F_{2k} - 0 = F_{2k}$$

More generally,

$$\int_{x_{2k}}^{x_{2n}} f_{m+1}(x)dx = F_{m-2k}e^{x_k} - F_{m-2n}e^{x_n} \tag{19}$$

Use of (15) and formulas for differentiation and integration lead to many others.

6. IDENTITIES

Many of the familiar identities for Fibonacci and Lucas numbers, besides (15), also have their counterparts for these Fibonacci and Lucas functions. Obtaining them is often merely a matter of substitution of

$$f_m(x) = \frac{\alpha^m e^{\alpha x} - \beta^m e^{\beta x}}{\sqrt{5}}, \quad l_m(x) = \alpha^m e^{\alpha x} + \beta^m e^{\beta x}$$

into one side of the identity, and the use of such relations as $\alpha + \beta = 1$, $\alpha\beta = -1$, $\alpha^2 + 1 = \alpha\sqrt{5}$, etc.

Thus, for example, one easily obtains

$$f_{m-1}(x)f_{m+1}(x) = f_m^2(x) + (-1)^m e^x \tag{20}$$

$$l_m(x) = f_{m-1}(x) + f_{m+1}(x) \tag{21}$$

$$5f_m^2(x) = l_m^2(x) + (-1)^{m-1}4e^x \tag{22}$$

$$f_{-m}(x) = (-1)^{m+1}e^x f_m(-x) \tag{23}$$

$$\left(\frac{l_m(x) + \sqrt{5}f_m(x)}{2}\right)^k = \frac{l_{km}(kx) + \sqrt{5}f_{km}(kx)}{2} \tag{24}$$

Note that the corresponding identities for the Fibonacci and Lucas <u>numbers</u> emerge immediately when $x = 0$.

From the formula (15) already treated come such familiar-appearing identities as

$$f_{2m-1}(2x) = f^2_{m-1}(x) + f^2_m(x) \quad \text{and} \quad f_{2m}(2x) = f_m(x)l_m(x) \tag{25}$$

$$f_{3m-1}(3x) = f^3_{m-1}(x) + 3f_{m-1}(x)f^2_m(x) + f^3_m(x)\ , \qquad \text{and} \tag{26}$$

$$f_{3m}(3x) = 3f^2_{m-1}(x)f_m(x) + 3f_{m-1}(x)f^2_m(x) + 2f^3_m(x)$$

while a generalization by induction on k and p yields

$$f_{km+p}(kx) = \sum_{i=0}^{k} \binom{k}{i} F_{i+p}f^{k-i}_{m-1}(x)f^i_m(x) \tag{27}$$

By using (15) to write

$$f_m(u)f_n(v) = (-1)\left[f_{m+1}(u)f_{n+1}(v) - f_{m+n+1}(u + v)\right]$$

and

$$(-1)f_{m+1}(u)f_{n+1}(v) = (-1)^2\left[f_{m+2}(u)f_{n+2}(v) - f_{m+n+3}(u + v)\right]$$

and adding, one obtains

$$f_m(u)f_n(v) = (-1)^2\left[f_{m+2}(u)f_{n+2}(v) - f_{m+n+2}(u + v)\right]$$

Repeating the process, and the use of induction lead to

$$f_m(u)f_n(v) = (-1)^r\left[f_{m+r}(u)f_{n+r}(v) - F_rf_{m+n+r}(u + v)\right] \tag{28}$$

Multiplication of (28) by $f_m(u)$ gives

$$f_m^2(u)f_n(v) = (-1)^r\left\{f_{m+r}(u)\left[f_m(u)f_{n+r}(v)\right] - F_r\left[f_m(u)f_{m+n+r}(u + v)\right]\right\}$$

while the use of (28) to expand the expressions in the square brackets here yields

$$(-1)^{2r}\left[f_{m+r}^2(u)f_{n+2r}(v) - 2F_r f_{m+r}(u)f_{m+n+2r}(u + v) + F_r^2 f_{2m+n+2r}(2u + v)\right]$$

whence induction leads to

$$(29) \qquad f_m^k(u)f_n(v) = (-1)^{kr}\sum_{i=0}^{k}\binom{k}{i}(-1)^i F_r^i f_{m+r}^{k-i}(u)f_{n+kr+im}(iu + v)$$

These two formulas are counterparts of two given by Halton [2]. In exactly the same way as he did, (29) can be used to develop a host of identities by choosing particular values of m, n, k, and r.

It is interesting to note that

$$(30) \qquad F_m f_m(v - u)e^u = (-1)^r\left[f_{m+r}(u)f_{n+r}(v) - f_r(u)f_{m+n+r}(v)\right]$$

is a "sibling" of (28), having been derived by substitution using (4), as a counterpart of the same formula

$$F_m F_n = (-1)^r\left[F_{m+r}F_{n+r} - F_r F_{m+n+r}\right] .$$

One is intrigued by the conjecture that they are both special cases of a more general formula in which no capital F's appear.

7. FIBONACCI FUNCTIONS OF TWO VARIABLES

Suppose $\{a_m\}$ is replaced by $\{f_m(x)\}$ in the series

$$\sum_{i=0}^{\infty} a_i \cdot \frac{y^i}{i!} \quad ,$$

to give a function of two variables $\phi(x, y)$.

$$\phi(x, y) = f_0(x) + f_1(x)y + f_2(x)\frac{y^2}{2!} + f_3(x)\frac{y^3}{3!} + \cdots \tag{31}$$

Differentiating term-by-term, we obtain

$$\frac{\partial\phi(x, y)}{\partial x} = f_1(x) + f_2(x)y + f_3(x)\frac{y^2}{2!} + f_4(x)\frac{y^3}{3!} + \cdots$$

$$\frac{\partial\phi(x, y)}{\partial y} = 0 + f_1(x) + f_2(x)\frac{2y}{2!} + f_3(x)\frac{3y^2}{3!} + f_4(x)\frac{4y^3}{4!} + \cdots$$

We see that

$$\frac{\partial\phi(x, y)}{\partial x} = \frac{\partial\phi(x, y)}{\partial y} \quad ,$$

and likewise it can be verified that all the second partial derivatives are the same, all the third partial derivatives are the same, etc. Let us therefore adopt the notation

$$\phi_0(x, y) = \phi(x, y), \quad \phi_1(x, y) = \frac{\partial\phi(x, y)}{\partial x} = \frac{\partial\phi(x, y)}{\partial y},$$

$$\phi_2(x, y) = \frac{\partial^2\phi(x, y)}{\partial x^2} = \frac{\partial^2\phi(x, y)}{\partial x\partial y} = \frac{\partial^2\phi(x, y)}{\partial y^2}, \cdots$$

so that

$$\phi_m(x, y) = \frac{\partial^m\phi(x, y)}{\partial x^r\partial y^s} = \sum_{i=0}^{\infty} f_{m+i}(x)\frac{y^i}{i!} = \sum_{i=0}^{\infty} f_{m+i}(y)\frac{x^i}{i!} \quad , \tag{32}$$

where r and s are positive integers such that $r + s = m$. Note that

(33) $$\phi_m(x, 0) = f_m(x), \quad \phi_m(0, y) = f_m(y), \quad \text{and} \quad \phi_m(0, 0) = F_m .$$

Expand $\phi_m(x, y)$ into a power series in two variables at $(0, 0)$:

$$\phi_m(x, y) = \phi_m(0, 0) + \left[x \frac{\partial \phi_m(0, 0)}{\partial x} + y \frac{\partial \phi_m(0, 0)}{\partial y} \right] + \frac{1}{2!}\left[x^2 \frac{\partial^2 \phi_m(0, 0)}{\partial x^2} + 2xy \frac{\partial^2 \phi_m(0, 0)}{\partial x \partial y} + y^2 \frac{\partial^2 \phi_m(0,0)}{\partial y^2} \right] + \cdots$$

$$= F_m + \left[xF_{m+1} + yF_{m+1} \right] + \frac{1}{2!}\left[x^2 F_{m+2} + 2xyF_{m+2} + y^2 F_{m+2} \right] + \cdots$$

$$= F_m + F_{m+1}\frac{(x+y)}{1!} + F_{m+2}\frac{(x+y)^2}{2!} + F_{m+3}\frac{(x+y)^3}{3!} + \cdots$$

Thus

(34) $$\phi_m(x, y) = f_m(x + y) = \frac{\alpha^m e^{\alpha(x+y)} - \beta^m e^{\beta(x+y)}}{\sqrt{5}}$$

8. REFERENCES

1. V. E. Hoggatt, Jr., "Fibonacci Numbers from a Differential Equation," Fibonacci Quarterly, Vol. 2, No. 3, October 1964, page 176.
2. John H. Halton, "On a General Fibonacci Identity," Fibonacci Quarterly, Vol. 3, No. 1, February 1965, pp. 31-43.

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