## A SHIFT FORMULA FOR RECURRENCE RELATIONS OF ORDER m

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It is well known that if  $F_i$  is the  $i^{th}$  Fibonacci number, then

$$\mathbf{F}_{n+k+1} = \mathbf{F}_{n+1}\mathbf{F}_{k+1} + \mathbf{F}_{n}\mathbf{F}_{k}$$

for all integers n, k. A generalization of this identity to recurrence relations of any order m is given here.

Let m be a positive integer and let  $p_1, p_2, \dots, p_m \quad (p_m \neq 0)$  be m elements of a field F. Furthermore, let  $\{y_i\}$  and  $\{U_i\}$  be two sequences in F obeying the recurrence relation whose auxiliary polynomial is

$$P(x) = x^m - \sum_{j=0}^{m-1} p_{m-j} x^j$$
,

and let  $\{U_i\}$  have the initial values

$$\mathbf{U}_0 = \mathbf{U}_1 = \cdots = \mathbf{U}_{m-2} = \mathbf{0}$$

and

$$U_{m-1} = 1$$
.

Then,

(1) 
$$y_{n+k} = \sum_{j=0}^{m-1} \sum_{i=0}^{j} p_{m-i} U_{k+i-j-i} y_{n+j}$$

for all integers n and k.

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The proof of (1) is by induction on k. Let n be fixed. For  $0 \le k < m$  it is clear that

(2) 
$$\sum_{i=0}^{j} p_{m-i} U_{k+i-j-1} = \begin{cases} 0 & \text{if } j < k \\ p_m U_{-1} = 1 & \text{if } j = k \\ \sum_{i=0}^{m-1} p_{m-i} U_{k+i-j-1} = U_{k+m-j-1} = 0 & \text{if } k < j < m \end{cases}$$

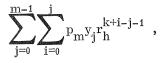
From (2) it immediately follows that (1) holds for  $k = 0, 1, \dots, m-1$ . From here, applications of the recurrence relation (corresponding to P(x)) for  $\{y_i\}$ and  $\{U_i\}$ , in both the forward and backward directions, easily prove that if (1) holds for  $k = h, h + 1, \dots, h + m - 1$ , then (1) holds for  $k = h-1, h, \dots, h + m$ . By application of finite induction, it follows that (1) holds for all integers n, k.

Let  $P(x) = (x - r_1)(x - r_2)\cdots(x - r_m)$  in an extension G of F and suppose that G is of characteristic zero. Further suppose that the  $r_j$  are pairwise distinct. Define  $D_k$  as the determinant produced by the process of substituting the vector  $(r_1^k, r_2^k, \cdots, r_m^k)$  for the m<sup>th</sup> row  $(r_1^{m-1}, r_2^{m-1}, \cdots, r_m^{m-1})$  in the Vandermonde determinant of  $r_1, r_2, \cdots, r_m$ . It is proven in [1] that for every integer k,

$$U_{k} = \frac{D_{k}}{D_{m-1}}$$

The case for repetitions among the  $r_j$  is handled in the following way: Start with the form for  $U_k$  in (3) and, pretending that the  $r_j$  are real, apply L'Hospital's Rule successively as  $r_I \rightarrow r_J$  for all repetitions  $r_I = r_J$  among the  $r_i^*$ .

A combination of (1) and (3) now comes with ease. Still taking the  $r_j$  to be pairwise distinct, define  $E_k$  as the determinant produced by the process of replacing the element  $r_h^k$  of the m<sup>th</sup> row of  $D_k$  by



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(3)

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and this for  $h = 1, 2, \dots, m$ . Then combination of (1) with (3) yields: For every integer k,

(4) 
$$y_k = \frac{E_k}{D_{m-1}}$$

The case for repeated roots is handled as with (3). In [2] identities akin to (4) are developed.

## REFERENCES

1. Arkin, Joseph, "Recurring Series," to appear in the Fibonacci Quarterly.

2. Styles, C. C., "On Evaluating Certain Coefficients," <u>The Fibonacci Quar</u>terly, Vol. 4, No. 2, April, 1966.

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