A RESULT FOR HERONIAN TRIANGLES

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Arising from some particular solutions communicated to me by Mr. W. W. Horner, I developed what seemed to be a new approach to the general problem of Heronian triangles. The results were interesting.

In such a triangle all three sides, and also the area, must be integral. Hence all three altitudes must be rational, as must be the sines of all three angles. It can be shown that the sides of such a triangle are divided into rational segments by the altitudes so that the cosines are also rational.

Now consider a Heronian triangle with sides a, b, c, with angle C contained by sides a and b.

Say, $\sin C = 2xy/(x^2+y^2)$, $\cos C = (x^2-y^2)/(x^2+y^2)$, where x and y are positive integers, $x > y$.

Using the cosine formula:

$$\cos C = (a^2+b^2-c^2)/2ab = (x^2-y^2)/(x^2+y^2)$$

So,

$$\begin{aligned}(x^2+y^2)c^2 &= (x^2+y^2)(a^2+b^2) - 2(x^2-y^2)ab\,.\\ \left[(x^2+y^2)c\right]^2 &= (x^2+y^2)^2a^2 - 2(x^4-y^4)ab + (x^2+y^2)^2b^2\\ &= \left[(x^2+y^2)a\right]^2 - 2(x^4-y^4)ab + (x^2-y^2)^2b^2 +\\ &\qquad + 4x^2y^2b^2\\ &= \left[(x^2+y^2)a - (x^2-y^2)b\right]^2 + (2xyb)^2\,,\end{aligned}$$

which has the fully general integral solution:

$$\left.\begin{aligned}(x^2+y^2)c &= (m^2+n^2)t\\ (x^2+y^2)a - (x^2-y^2)b &= (m^2-n^2)t\\ xyb &= mnt\end{aligned}\right\}\ \begin{array}{l}\text{m and n any positive integers,}\\ m > n.\ \text{And t a common ra-}\\ \text{tional divisor or multiplier.}\end{array}$$

Then

$$\left.\begin{aligned} xy(x^2 + y^2)a &= \left[xy(m^2 - n^2) + (x^2 - y^2)mn\right]t \\ xy(x^2 + y^2)b &= (x^2 + y^2)mnt \\ xy(x^2 + y^2)c &= xy(m^2 + n^2)t \end{aligned}\right\}$$

Without loss of generality, say $t = xy(x^2 + y^2)k$, then:

$$\left.\begin{aligned} a &= \left[xy(m^2 - n^2) + (x^2 - y^2)mn\right]k \\ b &= (x^2 + y^2)mnk \\ c &= xy(m^2 + n^2)k \end{aligned}\right\} \begin{array}{l}\text{where } k \text{ is any} \\ \text{rational common} \\ \text{divisor or multiplier.}\end{array}$$

The Heronian formula for area of a triangle is:

$$\Delta = \sqrt{s(s - a)(s - b)(s - c)}\ ,$$

where

$$2s = s + b + c\ .$$

Hence, substituting for a, b, c, we have:

$$\text{Area} = xymn(xm - yn)(xn + ym)k^2\ .$$

The results cover all Heronian triangles.

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A NOTE OF SADNESS

Mark Feinberg, a sophomore at the University of Pennsylvania, died Oct. 29, 1967, from injuries sustained in an automobile-motorcycle collision. It is a tragic loss to the Editorial Staff of the Fibonacci Quarterly Journal, as Mark had already published two articles in our pages. Included in this issue is a paper he last submitted.

This young scholar, Mark Feinberg, was both a brilliant young student and a winner of many prizes and scholarships. (Continued on page 490)