RECREATIONAL MATHEMATICS

JOSEPH S. MADACHY
4761 Bigger Road, Kettering, Ohio

DIGITAL DIVERSIONS

In the February 1968 issue of The Fibonacci Quarterly, I had asked readers to work at expressing Fibonacci numbers using the ten digits once only, in order, and using only the common mathematical operations and symbols. V. E. Hoggatt, Jr., the General Editor of this Journal, came up with a set of equations which, though not exactly what I had in mind, are of special interest because of their versatility. All ten digits are used and logarithms are required.

We start with

$$\log_2 2^n = n$$

or

$$\log_{\sqrt{\sqrt{\cdots\sqrt{2}}}} 2 = 2^n \quad (n \text{ radicals})$$

then

$$\log_2\left(\log_{\sqrt{\sqrt{\cdots\sqrt{2}}}} 2\right) = n \quad (n \text{ radicals})$$

This leads to

$$0 + \log_{(5-1)/2}\left[\log_{\sqrt{6/3}}(8-4)/(9-7)\right] = 1$$

or

$$0 + \log_{(5-1)/2}\left[\log_{\sqrt{\sqrt{\cdots\sqrt{6/3}}}}(8-4)/(9-7)\right] = n \quad (n \text{ radicals}) .$$

The study of all this eventually leads to the following:

$$\log_2 \left(\log_{\sqrt{\sqrt{\cdots\sqrt{m}}}} m\right) = n$$

(n radicals)

which further leads to the desired ten-digits-in-order form for any Fibonacci number, F_n :

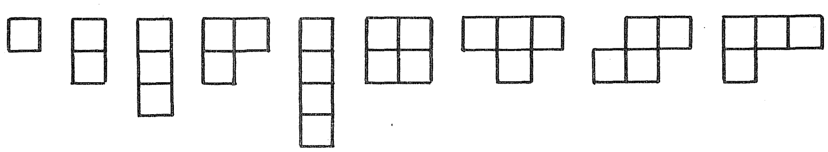
$$\log_{(0+1+2+3+4)/5} \left(\log_{\sqrt{\sqrt{\cdots\sqrt{6+7+8}}}} 9\right) = F_n \,.$$

(F_n radicals)

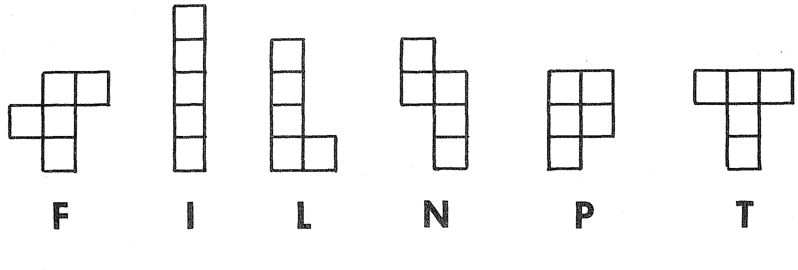
How about something more along these lines?

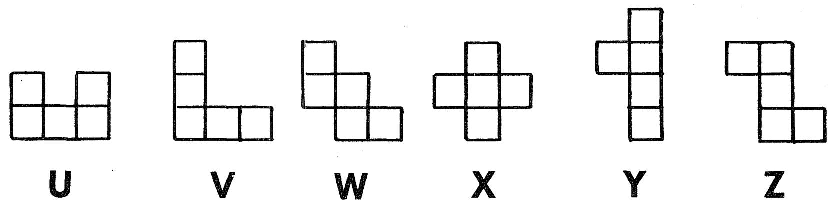
A PENTOMINO TILING PROBLEM

Ever since Solomon W. Golomb's article [1] appeared, much time has been devoted to the study of polyominoes and their properties. Polyominoes are configurations made up of squares connected edge-to-edge. The figures below show the first nine members of the polyomino family:

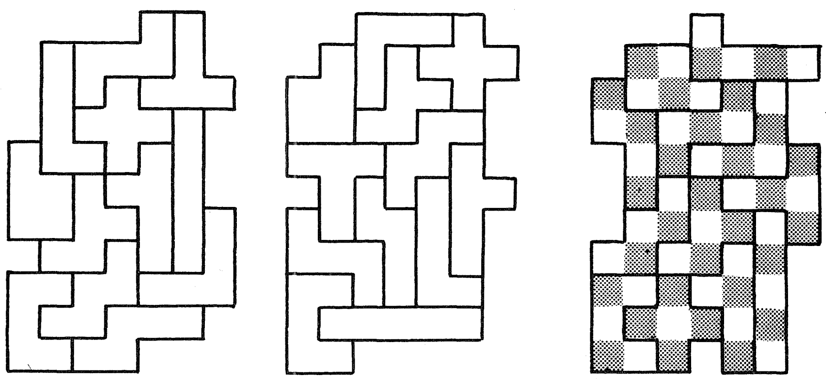


The first is a monomino, the second is a domino. The third and fourth figures are the two trominoes. The remaining figures are the five tetrominoes. Continued construction shows there are twelve pentominoes — those made with five squares. The pentominoes have proven so popular that they have had names assigned to them corresponding to their resemblance to certain letters of the alphabet. They are shown below.





Many polyomino problems have been posed, but here's a pentomino problem from Maurice J. Pova of Lancanshire, England: Find irregular patterns of the twelve pentominoes which form tessellation patterns; i. e., which cover a plane. There are 2339 distinct 6 x 10 rectangles which can be made from the pentominoes, but we are looking for irregular patterns. Three examples found by Povah are shown below. You should be able to find others.



The third figure has a bonus feature: the checkerboard pattern is maintained throughout the tessellation. The black and white squares fall on the same parts of each pentomino as it repeats in the plane.

ARE FIBONACCI NUMBERS "NORMAL"?

A "normal" number is one which contains the statistically expected number of each of the digits and combinations of digits. A random 100-digit number, if normal, ought to contain approximately 10 zeroes, 10 ones, 10 twos, and so on. For larger numbers, one could check for the expected occurrences of the pairs 10, 11, 12, 13, ..., 97, 98, 99. There is even a "poker hand" test for large enough numbers, in which groups of five digits are examined to see if the statistically expected number of "busts," "one pair," "full house," and other poker hands are present. Such a statistical study has been made of the digits of π [2].

I wondered if the Fibonacci numbers are normal. There are at least two ways of attacking the problem. The first method consists of examining each Fibonacci number and counting the number of distinct digits. By so doing I found some typical results.

F_n	Number of digits in F_n	Number of each of the following digits 0	1	2	3	4	5	6	7	8	9
F_{100}	21	1	3	3	1	3	3	1	2	2	2
F_{500}	105	9	8	19	8	7	11	11	11	11	10
F_{1000}	209	20	13	21	18	21	23	26	21	20	26

F_{100} is reasonably normal; F_{500} has more twos than expected; F_{1000} has a slightly low count of ones.

The second method consists of noting the cumulative sums of the digits. I did this up to F_{100} counting all the digits in all those 100 Fibonacci numbers. The results are tabulated below.

Number of each of the following digits to F_{100}

0	1	2	3	4	5	6	7	8	9
110	136	107	102	111	95	95	117	92	106

The total number of digits in the first 100 Fibonacci numbers is 1071. The distribution of the digits to F_{100} appears to be reasonably normal, except for the somewhat large number of ones.

Further work on this matter might lead to interesting speculation — depending on the results. The work of counting digits is tedious, but a computer could be programmed to calculate the Fibonacci numbers, count their digits, and print cumulative totals as well. Other statistical tests could be applied with the aid of a computer.

OBSERVATION

Has anyone noticed this before? While trying to see if the Fibonacci numbers could be used to make magic squares, I discovered that no set of consecutive Fibonacci numbers could be so used. Can you demonstrate this?

REFERENCES

1. Solomon W. Golomb, "Checkerboards and Polyominoes," Amer. Math. Monthly, Vol. 61, No. 10 (December 1954), pp. 675-682.

2. R. K. Pathria, "A Statistical Study of Randomness Among the First 10,000 Digits of π," Mathematics of Computation, Vol. 16, No. 78 (April 1962), pp. 188-197.

* * * * *

(continued from p. 191.)

$$\sum_{n=0}^{\infty} F_{n+k}^7 x^n = \frac{P_k(x)}{1 - 21x - 273x^2 + 1092x^3 + 1820x^4 - 1092x^5 - 273x^6 + 21x^7 + x^8}, \quad k=0, 1, 2, 3, 4, 5, 6, 7$$

$P_0(x) = x(1 - 20x - 166x^2 + 318x^3 + 166x^4 - 20x^5 - x^6)$

$P_1(x) = 1 - 20x - 166x^2 + 318x^3 + 166x^4 - 20x^5 - x^6$

$P_2(x) = 1 + 107x - 774x^2 - 1654x^3 + 1072x^4 + 272x^5 - 21x^6 - x^7$

$P_3(x) = 128 - 501x - 2746x^2 - 748x^3 + 1364x^4 + 252x^5 - 22x^6 - x^7$

$P_4(x) = 2187 + 32{,}198x - 140{,}524x^2 - 231{,}596x^3 + 140{,}028x^4 + 34{,}922x^5 - 2687x^6 - 128x^7$

$P_5(x) = 78{,}125 + 456{,}527x - 2{,}619{,}800x^2 - 3{,}840{,}312x^3 + 2{,}423{,}126x^4 + 594{,}364x^6 - 46{,}055x^6 - 2187x^7$

$P_6(x) = 2{,}097{,}152 + 18{,}708{,}325x - 89{,}152{,}812x^2 - 139{,}764{,}374x^3 + 85{,}906{,}864x^4 + 21{,}332{,}070x^5 - 1{,}642{,}812x^6 - 78{,}125x^7$

$P_7(x) = 62{,}748{,}417 + 483{,}369{,}684x - 2{,}429{,}854{,}358x^2 - 3{,}730{,}909{,}776x^3 + + 2{,}311{,}422{,}054x^4 + 570{,}879{,}684x^5 - 44{,}118{,}317x^6 - 2{,}097{,}152x^7.$

* * * * *