PERIODICITY AND DENSITY OF MODIFIED FIBONACCI SEQUENCES

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1. INTRODUCTION

Periodicity of the last digit (or last two digits and so on) in a Fibonacci sequence has been discussed by Geller [1], use being made of a digital computer, and solved theoretically by Jarden [2]. We may regard this as a periodic property of the right-most significant digit(s). There is a similar property for truncated Fibonacci sequences, the truncation being carried out prior to addition and on the right. Although this seems to be a somewhat artificial procedure it is the arithmetic involved on digital computers working in "floating point." The periodic property was noted by chance during a study of error propagation.

We generate a modified Fibonacci sequence from the recurrence

(1) $$u_n = u_{n-1} + u_{n-2} \quad (n = 2, 3, \cdots)$$

where for the moment u_0 and u_1 are arbitrary, but we retain only a certain number of left-most significant digits. To be more specific we work in an x-digit field $(x = 1, 2, \cdots)$ so that members of the sequence take the form

(2) $$u_n = n_1 n_2 n_3 \cdots n_x ,$$

where $n_j = 0, 1, \cdots, 9$ $(j = 1, 2, \cdots, x)$. In the addition of two such numbers

$$n_1 n_2 \cdots, n_x + N_1 N_2 \cdots, N_x$$

the sum is the ordinary arithmetic sum provided there is no overflow on the left; if there is an overflow then the sum is taken to be the first x digits from the left, the last digit on the right being discarded. In other words we are merely describing "floating point" arithmetic in frequent usage (to some base or other) on digital machines. For example, denoting the exponent by the symbol E,

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1-digit field

$$4 \text{ EO} + 5 \text{ EO} = 9 \text{ EO}$$
$$6 \text{ EO} + 7 \text{ EO} = 1 \text{ E1}$$

2-digit field

$$17 \text{ EO} + 82 \text{ EO} = 99 \text{ EO}$$
$$99 \text{ EO} + 9 \text{ EO} = 10 \text{ E1} .$$

Care is needed when the numbers being added do not belong to the same digit field. Thus

$$6 \text{ EO} + 1 \text{ E1} = 1 \text{ E1}$$
$$74 \text{ EO} + 14 \text{ E1} = 21 \text{ E1}$$

and so on. We confine our attention in this note to arithmetic to base ten and discuss some interesting and challenging properties of Fibonacci sequences in floating point arithmetic which have come to light after extensive work on an IBM 1620 computer.

2. CYCLE DETECTION AND PERIODIC PROPERTIES

One digit field

Take any two one-digit non-negative numbers (not both zero) and set up the modified Fibonacci sequence; then sooner or later the sequence invariably leads into the cyclic six-member set

(3) $$1, 1, 2, 3, 5, 8 .$$

For examples we have

(a) 3 EO, 6 EO, 9 EO, 1 E1, 1 E1, 2 E1, 3 E1, 5 E1, 8 E1.
(b) 4 EO, 1 EO, 5 EO, 6 EO, 1 E1, 1 E1, 2 E1, 3 E1, 5 E1, 8 E1.
(c) 1 EO, 0 EO, 1 EO, 1 EO, 2 EO, 3 EO, 5 EO, 8 EO, 1 E1.

It is convenient to drop the E-field symbol and indicate a change of E-field by a star. Thus (a) - (c) become

(A) 3, 6, 9, 1^*, 1, 2, 3, 5, 8 ;

(B) 4, 1, 5, 6, 1^*, 1, 2, 3, 5, 8;

(C) 1, 0, 1, 1, 2, 3, 5, 8, 1^* ;

where a change of field applies to all members of the sequence following a starred member. A proof of this cyclic property depends on two facts: first a Fibonacci sequence (modified or not) is determined if any two consecutive members are given, and second in view of the non-decreasing nature of the sequences, 1^* must occur with a non-zero predecessor thus leading into the cycle (if it occurred with a zero predecessor the cycle would already be established).

Two-digit field

For this there is the invariant 34-term cycle

10, 16, 26, 42, 68, 11^*, 17, 28, 45, 73, 11^*, 18,
29, 47, 76, 12^*, 19, 31, 50, 81, 13^*, 21, 34, 55,
89, 14^*, 22, 36, 58, 94, 15^*, 24, 39, 63 .

Reading by columns, a few examples are

37	45	74	02	04	91	18	56	77	99
21	64	00	91	04	19	16	93	34	50
58	10^*	74	93	08	11^*	34	14^*	11^*	14^*
79	16	74	18^*	12	12	50	23	14	19
13^*		14^*	27	20	23	84	37	25	33
20		21	45	32	35	13^*	60	39	52
33		35	72	52	58	21	97	64	85
53		56	11^*	84	93		15^*	10^*	13^*
86		91	18	13^*	15^*		24	16	21
13^*		14^*		21	24				
21		23							
		37							
		60							
		97							
		15^*							
		24							

sequences being terminated as soon as the cycle is joined.

x-digit field

Fields of length up to ten have been partially investigated with the following results:

Digit Field x	Cycle Length L(x)
1	6
2	34
3	139
4	67
5	3652
6	7455
7	79287
8	121567
9	1141412
10	4193114

Of course a completely exhaustive search for cycles is more or less out of the question; our search has involved some fifty or more cases with the four-digit field decreasing to less than five for the nine- and ten-digit fields. To say the least, the search in the fields of eight or more digits has been scanty; with this reservation in mind we remark that for the cycles so far found only the four-digit field yields different members in the 67-member cycle; in this case, there appear to be eight different cycles.

In passing we note that a modified Fibonacci sequence in an x-digit field must eventually repeat with cycle length less than 10^{2x}. For the sequence is determined by two consecutive members, and 10^{2x} is the number of different ordered pairs of x-digit numbers on base ten. Interest in the periodicity is heightened by the reduction in the observed cycle length as compared to the possible cycle length.

To identify the cycles the least number u_n and its successor u_{n+1} for the various fields x are as follows:

x	1	2	3	4	4	4	4	4	4	4	4
u_n	1	10	104	1004	1006	1010	1012	1015	1019	1026	1029
u_{n+1}	1	16	168	1625	1627	1634	1637	1642	1649	1660	1665

x	5	6	7	8	9	10
u_n	10002	103670	1616568	16167257	161803186	1618033864
u_{n+1}	16184	167741	2615662	26159171	261803054	2618033786

With these values the complete cycles can be generated without introducing alien members. It will be observed that the ratio u_{n+1}/u_n is near to its expected value $(1+\sqrt{5})/2 = 1.6180339885$ and increasingly so as the field length increases. In fact for the last six fields the ratio is as follows:

x	5	6	7	8	9	10
u_{n+1}/u_n	1.618076	1.618028	1.6180340	1.61803397	1.618033985	1.6180339887

Cycle Detection

Since members of a cycle beginning with a nine are far less common than for other leading digits, as we shall illustrate in the sequel, cycles are easiest to detect if a search is made for its largest members. Thus if we list the members beginning with nine and their successors, all we have to do is to generate a sequence until a matching pair appears. Cycle lengths are then readily picked up by sorting into order of magnitude the output of largest members at any given stage. The largest members in the various cycles we have found are

Field Length	Largest Member
1	8
2	94
3	958
4	9705, 9765, 9854, 9917
5	99810
6	999916
7	9999866
8	99998612
9	999998685
10	9999999229

3. FRACTION OF CYCLE WITH SPECIFIED LEADING DIGIT

An examination of the two-digit field cycle shows that 11 members have leading digit unity whereas only one member has leading digit nine. Is there an indication here of a general property? With this in mind an analysis of all the cycles available is given in Table 1.

Table 1

Fraction of Members of a Cycle with Stated Leading Digit for Different Fields

x = field length y = leading digit entry = corresponding fraction

x \ y	1	2	3	4	5	6	7	8	9
1	.33333	.16667	.16667	.00000	.16667	.00000	.00000	.16667	.00000
2	.32353	.17647	.11765	.08824	.08824	.05882	.05882	.05882	.02941
3	.30216	.17266	.12950	.09353	.07914	.07194	.05036	.05755	.04317
4	.29851	.17910	.13433	.08955	.07463	.07463	.05970	.04478	.04478
4	.31343	.16418	.13433	.08955	.08955	.05970	.05970	.04478	.04478
4	.29851	.17910	.11940	.10448	.07463	.05970	.05970	.05970	.04478
4	.29851	.17910	.11940	.08955	.08955	.05970	.05970	.05970	.04478
4	.29851	.17910	.11940	.10448	.07463	.07463	.05970	.04478	.04478
4	.29851	.17910	.11940	.10448	.07463	.07463	.05970	.04478	.04478
4	.29851	.17910	.13433	.08955	.07463	.07463	.05970	.04478	.04478
4	.29851	.17910	.11940	.10448	.07463	.07463	.04478	.05970	.04478
5	.30121	.17607	.12486	.09693	.07914	.06709	.05805	.05093	.04573
6	.30101	.17612	.12488	.09698	.07914	.06694	.05795	.05124	.04574
7	.30103	.17608	.12494	.09691	.07918	.06695	.05799	.05116	.04576
8	.30104	.17609	.12494	.09691	.07918	.06694	.05799	.05116	.04575
9	.30103	.17609	.12494	.09691	.07918	.06695	.05799	.05115	.04576

This table of fractional occurrences is of considerable interest. Notice that as the field size increases the fractional values become smoother for a given value of x. Moreover the fractions become closer to $\log_{10}(y + 1) - \log_{10}y$ as x increases. In fact we have

y	$\log_{10}(y + 1)/y$
1	.301030
2	.176091
3	.124939
4	.096910
5	.079181
6	.066947
7	.057992
8	.051153
9	.045758

For the nine-digit field the fractional values agree with those of the logarithmic difference to six decimal places excepting the two values for $y = 8, 9$, for which there is a discrepancy of one in the last decimal place.

It is interesting to recall that certain distributions of random numbers follow the "abnormal" logarithmic law. For example, it has been observed that there are more physical constants with low order first significant digits than high, and that logarithmic tables show more thumbing for the first few pages than the last. The interested reader in this aspect of the subject may care to refer to a paper by Roger S. Pinkham [3]. Pinkham remarks that the only distribution for first significant digits which is invariant under a scale change is $\log_{10}(y + 1)$. Following up the idea of the effect of a scale change we have taken each field cycle and multiplied the members by $k = 1, 2, \cdots, 9$ and compared the fractional occurrence of members with a given leading digit. A comparison over the k's for a particular field shows remarkable stability. The results of a field of five are given in Table 2. Results for larger fields show about the same stability.

5. CONCLUDING REMARKS

A number of interesting questions suggest themselves as follows:

(a) Is there an analytical tool which could be used to formulate the modified Fibonacci series for a specified field length? Perhaps one of the difficulties here, as pointed out by a referee, is the "one-way" nature of the sequences generated.

Table 2

Density of Members of Cycle According to Leading Digit
For Scaled-Up Field of Five

Scale Factor k Leading Digit y x = 5

y	1	2	3	4	5	6	7	8	9
k=1	.30093	.17606	.12486	.09693	.07913	.06709	.05778	.05148	.04573
2	.30120	.17606	.12486	.09693	.07913	.06709	.05778	.05148	.04545
3	.30093	.17634	.12486	.09693	.07913	.06681	.05805	.05120	.04573
4	.30093	.17606	.12513	.09693	.07913	.06709	.05778	.05120	.04573
5	.30093	.17606	.12486	.09721	.07941	.06681	.05805	.05093	.04573
6	.30093	.17606	.12486	.09721	.07913	.06709	.05778	.05120	.04573
7	.30093	.17606	.12486	.09666	.07941	.06709	.05805	.05120	.04573
8	.30093	.17606	.12486	.09693	.07913	.06681	.05832	.05120	.04573
9	.30120	.17579	.12513	.09666	.07913	.06681	.05805	.05148	.04573

(b) Have all the periods been found for fields of length up to $x = 10$? Are the period lengths the same for a given field length and are there cases similar to $x = 4$ in which there are several periods of the same length?

(c) Is there an asymptotic value for $1(x)$, the cycle length, when x is large?

(d) Is the fact that the density of occurrence of sequence members, with a specified leading digit, follows the so-called logarithmic law, when x is not small, trivial or significant?

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