# AMATEUR INTERESTS IN THE FIBONACCI SERIES III
# RESIDUES OF $u_n$ WITH RESPECT TO ANY MODULUS

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Dickson [1] reports that "J. L. Lagrange [2] noted that the residues of $A_k$ and $B_k$ with respect to any modulus are periodic." $A_k$ and $B_k$ are described by indicating that "Euler [3] noted that

$$(a+\sqrt{b})^k = A_k + B_k\sqrt{b}$$

implies

$$A_k = \tfrac{1}{2}[(a+\sqrt{b})^k + (a-\sqrt{b})^k], \quad B_k = \frac{1}{2\sqrt{b}}[(a+\sqrt{b})^k - (a-\sqrt{b})^k]"$$

With this as a hint I tried empirically to determine whether Lagrange's idea would work with the Fibonacci series, $u_n$. This may not be immediately apparent but simple empirical trials developed a number of significant revelations. Thus, starting with $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, etc., the residues for consecutive $u_n$, modulus 5 are: 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0. This series then repeats itself endlessly, illustrating Lagrange's periodicity. This is generally true of every modulus tried from 2 to 94. Each modulus has a characteristic period which displays various individual regularities. Thus, the above period, modulus 5, is broken up by zeros into 4 groups of 5 residues each including zero. The following is a resume of the characteristics of all groups and periods determined for all moduli investigated. We define

<u>Group</u>: The residues, starting with the residue from $u_1 = 1$ and continuing to and including the first zero residue obtained after dividing consecutive $u_n$.

<u>Period</u>: The residues, starting with the residue from $u_1 = 1$ and continuing to and including the first zero residue which follows a residue of 1, obtained after dividing consecutive $u_n$. From this point, the second period and all succeeding periods will exactly duplicate the first.

Examining the period of modulus 5 given above, from the above definitions, the first group comprises 5 digits, viz., 1, 1, 2, 3, 0. The period comprises 4 groups, containing 20 residues and ending 2, 2, 4, 1, 0.

The characteristics determined in the light of the above are:

1. The sum of all residues in a period (but not, in general, in a group), is divisible by the modulus without remainder. Thus, for modulus 5, the sum of the residues in the period is 40 which is divisible by the modulus.

2. The number of groups in a period is always 1, 2, or 4.

3. If the size of a group is n, then $u_n$ and more generally $u_{an}$ are exactly divisible by the modulus.

4. If $P_{n_1}$ and $P_{n_2}$ are prime factors of the modulus $P_{n_1}P_{n_2}$, the group and period of the modulus are divisible by the group and period respectively of the $P_n$'s. For example, modulus 10 is factored by $P_{n_1} = 2$ and $P_{n_2} = 5$. The group and period of 2 are 3 and the group and period of 5 are 5 and 20 respectively. The group size for modulus 10 is 15 (divisible by 3 and 5); the period, modulus 10, is 60, divisible by 3 and 20. This fact permits ready check of groups and period calculated for composite moduli.

It is evident that the finding listed as 3 above is not particularly helpful in determining the $u_n$ which a given prime modulus will divide, if the group size for that modulus must be determined by actual division of consecutive $u_n$. Thus, the prime 103 is found to have a group of n = 104. To determine that $u_{104}$ is divisible by 103 by dividing 104 consecutive $u_n$ and knowing that $u_{104}$ contains 22 digits, not to mention the large numbers which precede $u_{104}$, seems to be prohibitively laborious. Fortunately, early in the calculation of groups and periods I found a way to calculate these without any dividing at all! This was determined when it was noted that the residues are additive according to the usual Fibonacci series rule:

$$u_{n+2} = u_n + u_{n+1}$$

until the last residue is equal to or greater than the modulus.

At this time we subtract the modulus from this large residue. If the latter is equal to the modulus, the residue is zero and the group and/or period ends. If it is larger, the difference is set down as the residue in the place of the larger figure. This residue is then added to the previous residue and the sum is compared with the residue as before. This procedure continues until the group and/or period is determined. As can be seen, all manipulations are additions and subtractions, division is never required.

Example 1. To determine the group and period for modulus 10.

Start with $u_1 = 1$, the residue, $r_1 = 1$. Add this to $u_0 = 0$ and we get the second residue $r_2 = 1$. Add $r_1$ to

$$r_2 = 1 + 1 = r_3 = 2 .$$

This is still smaller than modulus 10, so we continue.

$r_2 + r_3 = 1 + 2 = r_4 = 3$. $r_3 + r_4 = 2 + 3 = r_5 = 5$. $r_4 + r_5 = 3 + 5 = r_6 = 8$.

Now,

$$r_5 + r_6 = 5 + 8 = 13 .$$

This is larger than modulus 10 so we subtract 10 and get $r_7 = 3$. Now we add

$$r_6 + r_7 = 8 + 3 = 11 .$$

Again, this is larger than the modulus; we subtract 10 and get $r_8 = 1$. Now we add

$r_7 + r_8 = 3 + 1 = r_9 = 4$. $r_8 + r_9 = 1 + 4 = r_{10} = 5$. $r_9 + r_{10} = 4 + 5 = r_{11} = 9$.

Now

$$r_{10} + r_{11} = 5 + 9 = 14 .$$

Subtract 10 and we have $r_{12} = 4$.

$$r_{11} + r_{12} = 9 + 4 = 13$$

from which $r_{13} = 3$.

$$r_{12} + r_{13} = 4 + 3 = r_{14} = 7 .$$

Finally

$$r_{13} + r_{14} = 3 + 7 = 10$$

which is exactly equal to the modulus. When we subtract 10 the result $r_{15} = 0$ and the group ends. Since $r_{14} \neq 1$, the period is not yet complete and is determined by continuing the procedure. Thus, listing consecutive residues starting with $r_{14}$ we get

7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0

ending the second group but the period continues:

9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0

Here the period ends, comprising 4 groups of 15 residues each. Notice that the second period begins exactly the same way as the first: 1, 1, 2, 3, etc. Since all periods are calculated the same way and all periods, regardless of modulus, start with 1, 1, it is obvious that all periods will be exact duplicates of each other and there is no point in continuing operations. Since the group size n, modulus 10, is 15 $u_{15}$ must be divisible by 10. We find $u_{15} = 610$, divisible by 10.

While it is evident that even this procedure is laborious for large prime numbers it is much easier than consecutive divisions of $u_n$. While short cuts such as this are possible in empirical investigations of the Fibonacci series, it is impossible to avoid labor altogether.

## REFERENCES

1. Dickson, History of the Theory of Numbers, 1, 1919, Chapter XVII, p. 393.
2. Cited in Dickson as: "Additions to Euler's Algebra 2, 1774, Sections 78-79, pp. 599-607, Euler, Opra Omnia (1), 1, 619."
3. Cited in Dickson as "Novi Connior. Acad. Petrop., 18, 1773, 185; Corum. Arith., 1, 554."

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