A LINEAR ALGEBRA CONSTRUCTED FROM FIBONACCI SEQUENCES PART II: FUNCTION SEQUENCES AND TAYLOR SERIES OF FUNCTION SEQUENCES

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In Part I, the algebra $\mathfrak{F}$ was constructed from the set of complex Fibonacci sequences. Finite polynomial and binomial interpretations were considered. We now consider a class of functions defined in $\mathfrak{F}$, which are models of prototype functions in C. These are extended to include Taylor series representations.

We first consider an auxiliary algebra, which is constructed from bits and pieces of easily recognizable structures. As in Part I, many of the proofs are elementary, and the reader is asked to fill in the details himself.

Definition 2.1 Let $G = \{(a, b): a, b \in C\}$, and define equality and three operations as follows: For $(a_1, a_2), (b_1, b_2) \in G$, $c \in C$,

1. $(a_1, a_2) = (b_1, b_2) \Longleftrightarrow a_1 = b_1, \quad a_2 = b_2$.
2. $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$.
3. $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$.
4. $c(a_1, a_2) = (ca_1, ca_2)$.

Theorem 2.1 G is a commutative linear algebra with unity (1,1).

Proof. The reader is asked to fill in the details.

Definition 2.2 Let $\phi: F \longrightarrow G$ be a function defined by the rule:

$$\phi(u_1, u_1) = (u_1 + \alpha u_1, \ u_0 + \beta u_1) \quad \text{for all } U \in \mathfrak{F}.$$

Theorem 2.2. $\phi: \mathfrak{F} \rightarrow G$ is an isomorphism.

Proof: ϕ is obviously a 1-1 linear transformation from the vector space $\mathfrak{F}$ onto the vector space G. We need only show that ϕ preserves multiplication. For $U, V \in \mathfrak{F}$,

$$
\begin{aligned}
(1)\qquad \phi(UV) &= \phi(u_0v_0 + u_1v_1,\ u_0v_1 + u_1v_0 + u_1v_1) \\
&= (u_0v_0 + u_1v_1 + \alpha(u_0v_1 + u_1v_0 + u_1v_1),\ u_0v_0 + u_1v_1 \\
&\qquad\qquad + \beta(u_0v_1 + u_1v_0 + u_1v_1)) \\
&= (u_0v_0 + \alpha(u_0v_1 + u_1v_0) + (\alpha + 1)u_1v_1\ ,\ u_0v_0 \\
&\qquad\qquad + \beta(u_0v_1 + u_1v_0) + (\beta + 1)u_1v_1) \\
&= ((u_0 + \alpha u_1)(v_0 + \alpha v_1), (u_0 + \beta u_1)(v_0 + \beta v_1)) = \phi(U)\phi(V)\ .
\end{aligned}
$$

The mapping ϕ was motivated by considering the linear factors of the characteristic number; i.e.,

$$C(U) = u_0^2 + u_0u_1 - u_1^2 = (u_0 + \alpha u_1)(u_0 + \beta u_1)\ .$$

Some fundamental vectors are mapped as follows:

1. $\phi(A) = \phi(1, \alpha) = (1 + \alpha^2, 0)$
2. $\phi(B) = \phi(1, \beta) = (0, 1 + \beta^2)$
3. $\phi(I) = \phi(1, 0) = (1, 1)$.

A, B determine the coordinate planes, and I determines a plane of symmetry, which will become significant later. A characteristic number for each

$$X = (x_1, x_2) \in G$$

can be defined as

$$C(X) = x_1 x_2\ .$$

Thus for $U \in \mathfrak{F}$, $C(U) = C(\phi(U))$.

Definition 2.3 Let f be an arbitrary function defined on a domain $D \subseteq C$. Define a corresponding $\hat{f}: D \times D \to G$ by the rule:

$$\hat{f}(X) = \hat{f}(x_1, x_2) = (f(x_1), f(x_2)).$$

whenever no confusion will exist, we will agree to identify $\hat{f}$ with f and write $\hat{f}(X) = f(X)$.

Definition 2.4. If f is defined on $D \subseteq C$, and if $U = (u_1, u_1) \in \mathfrak{F}$ is such that

$$u_0 + \alpha u_1, u_0 + \beta u_1 \in D,$$

define

$$f^{\star} : \phi^{-1}(D \times D) \longrightarrow F$$

by the rule:

$$f^{\star}(U) = \phi^{-1}(\hat{f}(X)),$$

where $X = \phi(U)$, or more simply

$$f^{\star}(U) = \phi \hat{f} \phi^{-1}(U) .$$

The notation used herein for composition of maps is: the order of events reads from left to right, or

$$\phi \hat{f} \phi^{-1}(U) = \phi^{-1}(\hat{f}(\phi(u))) .$$

We may again agree to identify $f^{\star}$ with $\hat{f}$ whenever no confusion will result, and say $f^{\star}(U) = \hat{f}(U) = f(U)$.

Theorem 2.3. The formula for $f^{\star}$ is

$$f^{\star}(U) = \frac{1}{\alpha - \beta}(\alpha^{-1}f(u_1 + \alpha u_1) - \beta^{-1}f(u_0 + \beta u_1) ,$$
$$f(u_0 + \alpha u_1) - f(u_0 + \beta u_1)).$$

Proof. The proof follows directly from Definition 2.4.

Corollary 2.1. If $f(x) = c$ (a constant), then $f^{\star}(U) = cI$.

Corollary 2.2 $f^{\star}(aI) = f(a)I$.

The reader may verify that the functions defined above are well-behaved Fibonacci sequences, and are thus, elements of $\mathfrak{F}$. The reader may further verify the following identities for some elementary functions: For $U, V \in \mathfrak{F}$,

1. $\exp U \exp V = \exp(U + V)$
2. $\exp(-U) = (\exp U)^{-1}$
3. $\sin^2 U + \cos^2 U = I$
4. $\sin U \cos U = \frac{1}{2} \sin 2U$
5. $\sin U (\cos U)^{-1} = \tan U.$

All operations must, of course, be those defined in $\mathfrak{F}$. The brute force approach required by Theorem 2.3 and the subsequent arithmetic in $\mathfrak{F}$ can be tempered by a trick: do the arithmetic in G.

Example 2.1. Show that

$$\sin(U + V) = \sin U \cos V + \cos U \sin V.$$

Since

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

is an identity in C, definition 2.1(1) gives

$$(2) \qquad (\sin(x_1 + y_1), \sin(x_2 + y_2)) = (\sin x_1 \cos y_1 + \cos x_1 \sin y_1, \; \sin x_2 \cos y_2 + \cos x_2 \sin y_2)$$

as an identity in G. We appeal now to definition 2.3 for the left side of (2) and to definition 2.1(2), (3) for the right side.

$$(3) \qquad \widehat{\sin}((x_1 + y_1), (x_2 + y_2)) = (\sin x_1, \sin x_2)(\cos y_1, \cos y_2) + (\cos x_1, \cos x_2)(\sin y_1, \sin y_2).$$

We now reverse our position and appeal to Definition 2.1 for the left side and Definition 2.3 for the right side of (3).

$$(4) \qquad \widehat{\sin}((x_1, x_2) + (y_1, y_2)) = \widehat{\sin}(x_1, x_2)\, \widehat{\cos}(y_1, y_2) + \widehat{\cos}(x_1, x_2)\, \widehat{\sin}(y_1, y_2),$$

$$\widehat{\sin}\,(X + Y) = \widehat{\sin}\,X\,\widehat{\cos}\,Y + \widehat{\cos}\,X\,\widehat{\sin}\,Y\,. \tag{5}$$

Definition 2.4, together with Theorem 2.2, yields

$$\sin^\star\,(U + V) = \sin^\star\,U\,\cos^\star\,V + \cos^\star\,U\,\sin^\star\,V\,. \tag{6}$$

the asterisk may be omitted because of our previous agreement.

We have proved in example 2.1 that

$$\sin\,(x + y) = \sin x \cos y + \cos x \sin y \in C \longrightarrow \sin\,(U + V) = \sin U \cos V + \cos V \sin V \in \mathfrak{F}\,.$$

Notice that, although the work was done in G, no element of G is evident in the final result. This is why G was called an auxiliary algebra in the introduction.

SOME SPECIAL FUNCTIONS

We could continue to define and explore Fibonacci function sequences ad infinitum, but we shall limit the discussion to two very elementary ones. First a theorem must be proved.

Theorem 2.4. If f and f^{-1} both exist on a subset of C, then

$$(f^\star)^{-1} = (f^{-1})^\star$$

on the corresponding subset of $\mathfrak{F}$.

Proof. $f^\star(U)$ is known from Theorem 2.3. Then

$$\begin{aligned} f^\star(U) \xrightarrow{\phi} \hat{f}(X) &= (f(x_1), f(x_2)) \xrightarrow{\widehat{f^{-1}}} (f^{-1}(f(x_1)), f^{-1}(f(x_2))) \\ &= (x_1, x_2) = X \xrightarrow{\phi^{-1}} U\,. \end{aligned} \tag{7}$$

From Definition 2.4, we have

$$(f^{-1}(f(x_1)), f^{-1}(f(x_2))) \xrightarrow{\phi^{-1}} (f^{-1})^\star\,(f^\star(U)\,)\,. \tag{8}$$

Since ϕ^{-1} is a mapping,

$$U = (f^{-1})^{\star}(f^{\star}(U)) ,$$

so

$$(f^{-1})^{\star} = (f^{\star})^{-1} .$$

A very fundamental function is now given by:

Definition 2.5. For $U, V \in \mathfrak{F}$, define $U^V = \exp(V \ln U)$.

When written in terms of the components of U, V,

$$U^V = \frac{1}{\alpha - \beta}\left(\alpha^{-1}(u_0 + \alpha u_1)^{v_0+\alpha v_1} - \beta^{-1}(u_0 + \beta u_1)^{v_0+\beta v_1},\right.$$

$$\left.(u_0 + \alpha u_1)^{v_0+\alpha v_1} - (u_0 + \beta u_1)^{v_0+\beta v_1}\right) . \tag{9}$$

Since $\ln z$ is a many valued function, some trouble may arise from Definition 2.5. The author offers the conjecture that no trouble will arise. Perhaps one of the readers will explore this possibility.

If $V = nI$, Definition 2.5 is specialized to Theorem 1.14. Another elementary but interesting set of relations are the multiple n^{th} roots of a sequence.

Theorem 2.5. There are n^2 distinct n^{th} roots of $U \neq 0 \in \mathfrak{F}$.

Proof. Let

$$r_1^n = |u_0 + \alpha u_1|, \; r_2^n = |u_0 + \beta u_1| ,$$

and

$$\alpha_i \quad (i = 0, 1, \cdots, n-1)$$

be the complex roots of unity. Then

$$U^{1/n} = \frac{1}{\alpha - \beta}\left(\alpha^{-1} r_1 \omega_i - \beta^{-1} r_2 \omega_j, \; r_1 \omega_i - r_2 \omega_j\right) . \tag{10}$$

If N is the number of possible solutions, then clearly $N \leq n^2$. We must show $N \nless n^2$. Assume the contrary; i.e., there are at least two identical solutions, which must be termwise equal.

(11)
$$\alpha^{-1}r_1\omega_i - \beta^{-1}r_2\omega_j = \alpha^{-1}r_1\omega_k - \beta^{-1}r_2\omega_\ell$$
$$r_1\omega_i - r_2\omega_j = r_1\omega_k - r_2\omega_\ell \ .$$

Both $\omega_i \neq \omega_k$ and $\omega_j \neq \omega_\ell$ must hold or the hypothesis is contradicted immediately. Thus,

(12)
$$\alpha r_2(\omega_j - \omega_\ell) = \beta r_1(\omega_i - \omega_k)$$
$$r_2(\omega_j - \omega_\ell) = r_1(\omega_i - \omega_k) \ .$$

If we substitute from the second equation into the first,

(13)
$$\alpha(\omega_i - \omega_k) = \beta(\omega_i - \omega_k) \ ,$$

which is clearly impossible unless $\omega_i = \omega_k$. This in turn implies that $\omega_j = \omega_\ell$. Thus, the hypothesis is contradicted, and the theorem is proved.

The reader is invited to find the four square roots of $F^2 = (1, 1)$ (cf. Theorem 1.12).

TAYLOR SERIES REPRESENTATIONS

In order to use the very useful concept of Taylor series representations of complex functions, a definition of convergence in $\mathfrak{J}$ must be formulated. A very short excursion into topology (metric spaces) will furnish the necessary foundation. Let d be the usual metric on C defined by

$$d(z_1, z_2) = |z_2 - z_1|$$

for all $z_1, z_2 \in C$. The next few theorems are so elementary that the proofs are omitted; however, they must be stated. Since the underlying set of G is $C \times C$, we may give

<u>Definition 2.6.</u> Let $\hat{d} : G \times G \longrightarrow R$ be defined by the rule:

$$\hat{d}(X, Y) = \max(d(x_1, y_1),\ d(x_2, y_2)) = \max(|y_1 - x_1|,\ |y_2 - x_2|) \ .$$

Theorem 2.6. $\hat{d}$ is a metric; hence, $(G, \hat{d})$ is a metric space.

An open sphere in G of radius r about the point X is

$$\hat{S}_r(X) = \{ Y \in G : \hat{d}(X, Y) \leq r \} .$$

If

$$\phi^{-1}(X) = U, \quad \phi^{-1}(Y) = V ,$$

then

$$\phi^{-1}(\hat{S}_r(X)) = S_r^\star(U) = \{V \in \mathfrak{F} : \max(|v_0 - u_0 + \alpha(v_1 - u_1)|, |v_0 - u_0 + \beta(v_1 - u_1)|) \leq r\}.$$

If we restrict $\mathfrak{F}$, G to real numbers, then

$$\phi^{-1}(\hat{S}_r(X))$$

is the interior of a golden rectangle with diagonal of length 2r, centered on U, and parallel to the vector I, and with short sides parallel to A, and long sides parallel to B. This fact should delight any true Fibonacciphile, and motivates:

Definition 2.7. Let $d^\star : \mathfrak{F} \times \mathfrak{F} \to R$ be defined by the rule:

$$d^\star(U, V) = \max(|v_0 - u_0 + \alpha(v_1 - u_1)|, |v_0 - u_0 + \beta(v_1 - u_1)|) .$$

Theorem 2.7. $d^\star$ is a metric; hence, $(F, d^\star)$ is a metric space.

Theorem 2.8. $\phi : (\mathfrak{F}, d^\star) \to (G, \hat{d})$ is a homeomorphism.

By design the metric spaces $(\mathfrak{F}, d^\star)$, $(G, \hat{d})$ are topologically equivalent. The necessary groundwork has now been laid for the theorem on convergence.

Theorem 2.9. If

$$f(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^i$$

is a Taylor series for $z \in S_r(z_0)$, then

$$f^\star(U) = \sum_{i=0}^{\infty} a_i(U - z_0I)^i \quad \text{for} \quad U \in S_r^\star(z_0I) .$$

Furthermore:

$$(f^\star(U))_k = \sum_{i=0}^{\infty} a_i((U - z_0I)^i)_k , \qquad k = 0, 1, \cdots .$$

Proof. Let $z_1, z_2 \in S_r(z_0)$. Then for any $\epsilon > 0$, there are N_1, N_2 such that for $n > \max(N_1, N_2)$, we have

$$\sum_{i=0}^{n} a_i \,(z_1 - z_0)^i \in S_\epsilon(f(z_1)) , \quad \text{and} \tag{14}$$

$$\sum_{i=0}^{n} a_i \,(z_2 - z_0)^i \in S_\epsilon(f(z_2)) . \tag{15}$$

Since these sums are in the coordinate spaces of G, we have

$$\sum_{i=0}^{n} a_i((z_1 - z_0)^i , (z_2 - z_0)^i) \in \hat{S}_\epsilon(f(z_1), f(z_2)) . \tag{16}$$

But by the definitions of operations in G,

$$\begin{aligned} ((z_1 - z_0)^i, (z_2 - z_0)^i) &= (z_1 - z_0, z_2 - z_0)^i \\ &= ((z_1, z_2) - (z_0, z_0))^i \\ &= (Z - Z_0)^i \quad \text{for} \quad i = 0, 1, \cdots . \end{aligned} \tag{17}$$

Hence,

$$\sum_{i=0}^{n} a_i (Z - Z_0)^i \in \hat{S}_\epsilon (f(Z)) . \tag{18}$$

Let $U = \phi^{-1}(Z)$. Then

$$U \in \phi^{-1}(\hat{S}_r(Z_0))$$

or

$$U \in S_r^\star (z_0 I) .$$

By Theorem 2.8,

$$\sum_{i=0}^{n} a_i (U - z_0 I)^k \in S_\epsilon^\star (f^\star(U)) . \tag{19}$$

Since $C \times C$ is the underlying set of $\mathfrak{F}$ and G, and since $C \times C$ is always complete as a metric space, the limits exist, which proves the first statement of the theorem.

Now consider a partial sum with remainder in G.

$$f(Z) - \sum_{i=0}^{n} a_i (Z - Z_0)^i = (e_1, e_2) . \tag{20}$$

Since this is a finite sum, write the k^{th} term under the mapping ϕ^{-1}.

$$(f(U))_k - \sum_{i=0}^{n} a_i ((U - z_0 I)^i)_k = (e_1, e_2)_k = (E)_k . \tag{21}$$

From the first part of the proof, $E \to O$, and by definition $(O)_k = 0$. Hence $(E)_k \to 0$ for each k and the theorem is proved.

Example 2.2. Let

$$(1 - z)^{-(k+1)} = \sum_{i=0}^{\infty} \binom{k+i}{k} z^i \quad \text{for } z \in S_1(0) \tag{22}$$

be the prototype complex function. Clearly $\frac{1}{2}F \in S_1^{\star}(0)$. By Theorem 2.9 we may write

$$(I - \tfrac{1}{2}F)^{-(k+1)} = \sum_{i=0}^{\infty} \binom{k+i}{k} (\tfrac{1}{2}F)^i \, . \tag{23}$$

Reducing the left side of equation 23 yields

$$\begin{aligned} (I - \tfrac{1}{2}F)^{-(k+1)} &= (((1,0) - (0,\tfrac{1}{2}))^{-1})^{k+1} \\ &= (2(2,\ -1)^{-1})^{k+1} = 2^{k+1} F^{2k+2} \, . \end{aligned} \tag{24}$$

Taking the $(j+1)^{st}$ term from each side of equation 23 gives

$$2^{k+1} F_{2k+2+j} = \sum_{i=0}^{\infty} \binom{k+i}{k} \frac{F_{i+j}}{2^i}\,, \qquad \begin{aligned} k &= 0, 1, 2, \cdots, \\ j &= 0, \pm 1, \pm 2, \cdots \, . \end{aligned} \tag{25}$$

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