# THE TWIN PRIME PROBLEM AND GOLDBACH'S CONJECTURE IN THE GAUSSIAN INTEGERS

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## 1. PRELIMINARIES

The set of Gaussian Integers denoted by G is the set $\{a + bi\}$, a and b real integers and i the "imaginiary unit." It is well known that G with the usual two operations is an integral domain and that the division algorithm holds, if for any $\alpha$ and $\beta \neq 0$ in G, there are $\gamma$ and $\delta$ in G such that $\alpha = \beta\gamma + \delta$, where $|\delta| \leq |\beta|$. Since the division algorithm holds on G, the domain is a unique factorization domain.

A Gaussian prime is a Gaussian integer, $\rho$, such that:

i) $|\rho| \geq 1$ and

ii) if $\alpha$ divides $\rho$ then $|\alpha| = 1$ or $\alpha = \epsilon\rho$ where $\epsilon$ in G and $|\epsilon| = 1$. Here divides means that if $\alpha$ divides $\beta$. then there is a Gaussian Integer $\gamma$ such that $\alpha\gamma = \beta$.

The Gaussian Primes can be separated into the following three classes:

i) if p is a positive real prime of the form $4k + 3$, the $\pm p$ and $\pm ip$ are Gaussian primes.

ii) if p is a positive real prime of the form $4k + 1$, the p can be ex— pressed uniquely as $p = a^2 + b^2$ and the expression generates the 8 Gaussian Primes $\pm a \pm bi$ and $\pm b \pm ai$.

iii) $\pm 1 \pm i$ are Gaussian Primes.

A Gaussian integer, $\beta$, is said to be even if $1 + i$ divides $\beta$. An easy method of recognizing even Gaussian Integers is the following:

A Gaussian Integer, $\beta = a + bi$, is even if, and only if, 2 divides $a + b$ or in other words, if a and b have the same parity.

Consider the figure which plots the Gaussian Primes in the square with vertices at $\pm 50 \pm 50i$.

## 2. TWIN PRIMES

A meaningful definition is sought for twin primes in the Gaussian Integers. We have a preference for the following, Definition: Two Gaussian Primes $\rho_1$ and $\rho_2$ are called Gaussian twin primes if $\rho_1 - \rho_2 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

Our reason for preferring this definition is that $\pm 1 \pm i$ are the only even primes, and in the real case primes are twins if their differences are $\pm 2$.

Notice in the figure the relative scarcity of primes that are not twins, the smallest odd ones being $17 \pm 12i$ and their associates. It is perhaps coin— cidence but $|17 + 12i| = 20.8+$, which is fairly close to 23, which is the small— est odd real primes, which is not a twin. Notice that 23 and $24+i$ are twin Gaussian Primes and that 47 is not a twin in either system. This serves to point out that there is little, if any, connection between numbers being real twin primes and being Gaussian Twin Primes.

There are two possibilities of definitions for triplets of primes in the Gaussian Integers. The most natural seems to us to be

Definition 2. Three Gaussian Primes, $\rho_1, \rho_2, \rho_3$ are called Gaussian triplet primes if $\rho_1 - \rho_2 = \rho_2 - \rho_3 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

An example of these triplets would be $20 + 3i$, $21 + 4i$, and $22 + 5i$. The alternate definition would be for the less restrictive condition on the 's: $|\rho_1 - \rho_2| = |\rho_2 - \rho_3| = |1 + i|$. Examples of this less restrictive condition for the triplets would be

(2A) $10 + i$, $11$, and $10 - i$

(2B) $19 + 10i$, $20 + 11i$, and $21 + 10i$.

The only real primes that could be considered triplets would be 3, 5, and 7. But it can be noticed from the figure that there are many Gaussian triplet primes.

There are also several possibilities for definitions for Gaussian quadruplet primes. The one we prefer is the more restrictive.

Definition 3: Four Gaussian primes, $\rho_1, \rho_2, \rho_3$, and $\rho_4$ are Gaussian quadruplet primes if $\rho_1 - \rho_2 = \rho_2 - \rho_3 = \rho_3 - \rho_4 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

Two examples of these are the primes $31 + 26i$, $32 + 27i$, $33 + 28i$, $34 + 29i$; and $16 + 19i$, $17 + 18i$, $18 + 17i$, and $19 + 16i$.

The less restrictive definition would have

$$|\rho_1 - \rho_2| = |\rho_2 - \rho_3| = |\rho_3 - \rho_4| = |1 + i| .$$

This would not only allow the first definition, but would allow forms like

(3A) $25 + 12i$, $26 + 11i$, $27 + 10i$, and $25 + 9i$, or

(3B) $49 + 34i$, $48 + 35i$, $49 + 36i$, and $48 + 35i$ or

(3C) $24 + 5i$, $25 + 4i$, $26 + 5i$, and $25 + 6i$.

A further loosening might be imposed on the restrictions to allow forms like $43 + 10i$, $44 + 9i$, $45 + 8i$, and $43 + 8i$ by making the condition in the definition that for some j $|\rho_k - \rho_j| = |1 + i|$, for $k \neq j$.

The most restrictive definition for quintuplets would be

Definition 4: The Gaussian Primes $\rho_1$, $\rho_2$, $\rho_3$, $\rho_4$, and $\rho_5$ are Gaussian quintuplet primes if $\rho_1 - \rho_2 = \rho_2 - \rho_3 = \rho_3 - \rho_4 = \rho_4 - \rho_5 = (1 + i)\epsilon$ where $|\epsilon| = 1$.

Several less restrictive definitions could be posed that would allow a variety of forms such as the zigzag: $13 + 2i$, $14 + i$, $15 + 2i$, $16 + i$, and $17+ 2i$. We do not wish to list examples of these forms.

We do wish to notice the following:

Theorem 1: There are only finitely many Gaussian quintuplet primes and they are $\pm5\pm2i$, $\pm4\pm i$, $\pm3$, $\pm2\pm i$, $\pm1\pm2i$, $\pm3i$, $\pm1\pm4i$, $\pm2\pm5i$.

Proof: A special division algorithm for $2 + i$ asserts that for any Gaussian Integer $\gamma$, there are Gaussian Integers $\alpha$ and $\beta$ such that $\gamma = \alpha(2 + i) + \beta$ with $|\beta| \leq 1$, hence $\beta = 0$ or $\pm1$ or $\pm i$. (See representation C of [1] for details.)

Now consider

$\rho_1 = \alpha(2 + i) + \beta$ with $|\beta| \leq 1$, and suppose that the $\epsilon$ in the theorem is $-1$, then

i) if $\beta = 0$ then $(2 + i)\alpha = \rho_1$

ii) if $\beta = 1$ then $\rho_2 = (2 + i)(\alpha + 1)$

iii) if $\beta = i$ then $\rho_4 = (2 + i)(\alpha + 2 + i)$

iv) if $\beta = -1$ then $\rho_5 = (2 + i)(\alpha + 2 + i)$

v) if $\beta = -i$ then $\rho_3 = (2 + i)(\alpha + 1)$

So in any case, $2 + i$ is a factor of one of the $\rho_j$'s. Hence at least one of the $\rho_j$'s is composite unless the $\rho_j$ is $(2 + i)\delta$, where $|\delta| = 1$. This only happens when the $\rho_j$'s are in the set specified in the theorems. Similar arguments can be given for $\epsilon = 1$ or $\pm i$.

There are several other definitions that could arise, but we choose to start guessing about the definitions we now have:

Conjecture A: There are infinitely many Gaussian twin primes.

Conjecture B: There are infinitely many Gaussian triplet primes.

Conjecture C: There are infinitely many Gaussian quadruplet primes.

It is clear that if conjecture C is true then the others would be true, and if conjecture A is false, then the others would be false. It seems to us that all three should be either true or false together, but this is only our opinion.

One theorem that can be stated positively about the primes that form a square of twin primes like those of example 3C is the following:

Theorem 2: If $a \pm 1 + bi$, $a + (b \pm 1)i$ are primes with $|a| + |b| > 5$, then $a$ and $b$ are both multiples of 5 and neither is zero.

Proof: Since these four numbers are primes, none is divisible by $2 + i$ nor $2 - i$. The strong division algorithm for $2 + i$ gives $a + bi = (2 + i)\alpha + \delta$ where $|\delta| \leq 1$. But if $\delta = 1$, then $(a - 1) + bi = (2 + i)\alpha$; if $\delta = i$, then $a + (b - 1)i = (2 + i)\alpha$; if $\delta = -1$, then $a + 1 + bi = (2 + i)\alpha$; and if $\delta = -i$, then $a + (b + 1)i = (2 + i)\alpha$ so $\delta = 0$. A similar argument implies that for $a + bi = (2 + 1)\beta + \eta$ then $\eta = 0$. So not only $2 + i$ but also $2 - i$ divides $a + bi$; hence $(2 + i)(2 - i) = 5$ divides $a + bi$, hence 5 divides each component $a$ and $b$.

Notice that if $b = 0$ then $a + 1$ and $a - 1$ are both primes which is impossible because if $a + 1$ is even, 2 divides it, and if $a + 1$ and $a - 1$ are both odd, one is of the form $4k + 1$, which is not a Gaussian prime. A similar argument settles the case $a = 0$.

Corollary. If $\rho_1$, $\rho_2$, $\rho_3$, and $\rho_4$ are a set of Gaussian primes as described in theorem 2, then there does not exist a Gaussian prime $\rho \neq \rho_j$ such that $\rho = \rho_j + (1 + i)\epsilon$ for $|\epsilon| = 1$.

Proof. Notice that the eight odd numbers that surround this set have the property that they differ from $a + bi$ by $\pm 2 \pm i$ or $\pm 1 \pm 2i$ hence are divisible by either $2 + i$ or $2 - i$ since $a + bi$ is divisible by both.

This means that forms like 3C that are not near the origin can not have an additional prime attached a checker move away.

## 3. GOLDBACH'S CONJECTURE

There are several possibilities for generalizing Goldbach's conjecture. One possibility would be

Conjecture D: If $\alpha$ is an even Gaussian Integer, then there are Gaussian Primes $\rho_1$ and $\rho_2$ such that $\alpha = \rho_1 + \rho_2$.

This seems to us to be a poor generalization of Goldbach's conjecture. It is more the generalization of the statement, "Every even integer is either the sum or difference of two positive primes."

Since positive is meaningless in the Gaussian Integers, we would like to somehow purge the possibility of allowing differences to creep in. These two possibilities occur: 1) Insist the $\rho_1$ and $\rho_2$ lie in that same half plane, or 2) insist that $|\rho_1|$ and $|\rho_2|$ be $\leq |\alpha|$. We however prefer this one.

Conjecture E: If $\alpha$ is an even Gaussian Integer with $|\alpha| > \sqrt{2}$, then there are Gaussian Primes, $\rho_1$ and $\rho_2$, such that $\alpha = \rho_1 + \rho_2$ and the angles $\rho_1 0 \alpha$ and $\alpha 0 \rho_2$ are $\leq 45°$.

It is easy to see that conjecture E implies conjecture D and both of the two alternatives mentioned.

Conjecture E has been verified for all even Gaussian Integers in the figure.

Certain conditions stronger than conjecture E might be asserted by reducing 45°. The $\alpha$ may have to be increased in absolute value to avoid certain exceptional cases. For example

Conjecture F: If $\alpha$ is an even Gaussian Integer with $|\alpha| > \sqrt{10}$, then there are primes $\rho_1$ and $\rho_2$ with angles $\rho_1 0 \alpha$ and $\alpha 0 \rho_2 \leq 30°$ and $\alpha = \rho_1 + \rho_2$.

This has also been verified for the even integers in the figure. Note that $1 + 3i$, $3 + i$, and $2$ and their associates require 45°.

Reducing the angle to 0° doesn't work since 4, 8, 12 $\cdots$ have no representatives as the sum of two Gaussian Primes. There might be some sacred angle $\theta$, which is the dividing point for the truth or falsity of the appropriate conjecture or perhaps if $\theta > 0$ then for all $|\alpha| > N_\theta$ the appropriate conjecture may be true. There might be a universal shaped region that depends on $|\alpha|$ such that the primes $\rho_1$ and $\rho_2$ would fall in that region, with this region in some ways minimal.

(Continued on p. 92.)