ON THE LINEAR DIFFERENCE EQUATION WHOSE SOLUTIONS ARE THE PRODUCTS OF SOLUTIONS OF TWO GIVEN LINEAR DIFFERENCE EQUATIONS

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It was shown by Appell [1] that if u_1 and u_2 denote two linearly independent solutions of

$$\left\{D^2 + p(t)D + q(t)\right\}y = 0 ,$$

then u_1^2, u_1u_2, u_2^2 denote three linearly independent solutions of the third-order linear differential equation

$$\left\{D^3 + 3pD^2 + (2p^2 + p' + 4q)D + (4pq + 2q')\right\} y = 0 .$$

Watson [2] shows that if

$$\left\{D^2 + I\right\}v = 0, \quad \left\{D^2 + J\right\}w = 0 ,$$

then $y = vw$ satisfies the fourth-order differential equation

$$D\left\{\frac{y'' + 2(I + J)y' + (I' + J')y}{I - J}\right\} = -(I - J)y, \quad (I \neq J) .$$

Bellman [3] gives a matrix method for obtaining Appell's result and notes that the method can be used to find the linear differential equation of order mn whose solutions are the products of the solutions of a linear differential equation of order m and one of order n.

We now obtain analogous results for linear difference equations.

Let $\left\{A_n\right\}$ and $\left\{B_n\right\}$ denote sequences defined by the second-order linear difference equations

(1) $$A_{n+1} = P_nA_n + Q_nA_{n-1} ,$$

(2) $$B_{n+1} = R_nB_n + S_nB_{n-1} ,$$

where A_0, A_1, B_0, B_1 are arbitrary and P_n, Q_n, R_n, S_n are given.

If u_1 and u_2, v_1 and v_2 denote pairs of linearly independent solutions of (1) and (2), respectively, then we first obtain the third-order linear difference equation whose solution is given by

$$k_1u_1^2 + k_2u_1u_2 + k_3u_2^2$$

where the k_i's are constants. Squaring (1) and letting $C_n = A_n^2$, we obtain

$$C_{n+1} = P_n^2C_n + Q_n^2C_{n-1} + 2P_nQ_nA_nA_{n-1} \tag{3}$$

or

$$\begin{aligned} C_{n+1} - P_n^2C_n - Q_n^2C_{n-1} &= 2P_nQ_nA_{n-1}(P_{n-1}A_{n-1} + Q_{n-1}A_{n-2}) \\ &= 2P_nP_{n-1}Q_nC_{n-1} + 2P_nQ_nQ_{n-1}A_{n-1}A_{n-2} \ . \end{aligned} \tag{4}$$

By decreasing the index n by 1 in (3), we can eliminate $A_{n-1}A_{n-2}$ to obtain

$$\begin{aligned} P_{n-1}C_{n+1} = P_n(P_nP_{n-1} + Q_n)C_n &+ P_{n-1}Q_n(P_nP_{n-1} + Q_n)C_{n-1} \\ &- P_nQ_nQ_{n-1}^2C_{n-2} \end{aligned} \tag{5}$$

We now obtain the fourth-order equation whose solution is given by $k_1u_1^3$ $+ k_1u_1^3 + k_2u_1^2u_2 + k_3u_1u_2^2 + k_4u_2^3$. Cubing (1) and letting $D_n = A_n^3$, we obtain

$$\begin{aligned} D_{n+1} - P_n^3D_n - Q_n^3D_{n-1} &= 3P_n^2Q_nA_n^2A_{n-1} + 3P_nQ_n^2A_nA_{n-1}^2 \\ &= 3P_n^2Q_nA_n^2A_{n-1} + 3P_nQ_n^2A_{n-1}^2(P_{n-1}A_{n-1} + Q_{n-1}A_{n-2}) \end{aligned} \tag{6}$$

or

$$\begin{aligned} D_{n+1} - P_n^3D_n - Q_n^2(3P_nP_{n-1} + Q_n)D_{n-1} &= 3P_n^2Q_nA_n^2A_{n-1} \\ + 3P_nQ_n^2Q_{n-1}A_{n-1}^2A_{n-2} &= 3P_n^2Q_nA_{n-1}(P_{n-1}A_{n-1} + Q_{n-1}A_{n-2})^2 \\ + 3P_nQ_n^2Q_{n-1}A_{n-1}^2A_{n-2} & \end{aligned} \tag{7}$$

or

$$(8) \qquad D_{n+1} - P_n^3 D_n - Q_n(3P_nP_{n-1}Q_n + 3P_n^2P_{n-1}^2 + Q_n^2)D_{n-1}$$
$$= 3P_nQ_nQ_{n-1}(2P_nP_{n-1} + Q_n)A_{n-1}^2A_{n-2} + 3P_n^2Q_nQ_{n-1}^2A_{n-2}^2A_{n-1} .$$

By reducing the index of n by 1 in (6), we can then solve (6) and (8) for $A_{n-1}^2A_{n-2}$. Then by substituting this expression in (7), we can obtain the desired difference equation.

To find the fourth-order equation satisfied by

$$k_1u_1v_1 + k_2u_1v_2 + k_3u_2v_1 + k_4u_2v_2 ,$$

we multiply (1) by (2) and let $E_n = A_nB_n$, to give

$$(9) \qquad E_{n+1} - P_nR_nE_n - Q_nS_nE_{n-1} = P_nS_nA_nB_{n-1} + R_nQ_nB_nA_{n-1}$$
$$= P_nS_nB_{n-1}(P_{n-1}A_{n-1} + Q_{n-1}A_{n-2})$$
$$+ R_nQ_nA_{n-1}(R_{n-1}B_{n-1}' + S_{n-1}B_{n-3})$$

or

$$(10) \qquad E_{n+1} - P_nR_nE_n - (P_nP_{n-1}S_n + R_nR_{n-1}Q_n + Q_nS_n)E_{n-1} = P_nS_nQ_{n-1}B_{n-1}A_{n-2}$$
$$+ R_nQ_nS_{n-1}A_{n-1}B_{n-2} = P_nS_nQ_{n-1}A_{n-2}(R_{n-2}B_{n-2} + S_{n-2}B_{n-3})$$
$$+ R_nQ_nS_{n-1}B_{n-2}(P_{n-2}A_{n-2} + Q_{n-2}A_{n-3})$$

or

$$(11) \qquad E_{n+1} - P_nR_nE_n - (P_nP_{n-1}S_n + R_nR_{n-1}Q_n + Q_nS_n)E_{n-1}$$
$$- (P_nS_nQ_{n-1}R_{n-2} + R_nQ_nS_{n-1}P_{n-2})E_{n-2}$$
$$= P_nQ_{n-1}S_nS_{n-2}A_{n-2}B_{n-3} + R_nS_{n-1}Q_nQ_{n-2}B_{n-2}A_{n-3} .$$

By now reducing the index n by 2 in (9) and by 1 in (10), we can then eliminate $A_{n-2}B_{n-3}$ and $B_{n-2}A_{n-3}$ from (9), (10), and (11), to obtain the desired difference equation.

If here P_n, Q_n, R_n, S_n are independent of n, the equations simplify and the elimination is rather simple. This special case gives a solution to part (i) of proposed problem H-127 by M. N. S. Swamy (Fibonacci Quarterly, Feb., 1968, p. 51), i. e., "The Fibonacci polynomials are defined by

$$f_{n+1}(x) = xf_n(x), \quad n \geq 2 ,$$
$$f_1(x) = 1 \quad \text{and} \quad f_2(x) = x .$$

If $z_r = f_r(x)f_r(y)$, then show that (i) z_r satisfies the recurrence relation

$$z_{n+4} - xyz_{n+3} - (x^2 + y^2 + 2)z_{n+2} - xyz_{n+1} + z_n = 0. \text{"}$$

We now extend Bellman's matrix method, with little change, to difference equations.

First we give an analogous lemma for difference equations.

Lemma. Let Y and Z denote, respectively, the solutions of the matrix difference equations

$$EY = A(n)Y , \quad Y(0) = I ,$$
$$EZ = ZB(n) , \quad Z(0) = I ,$$

then the solution of

$$EX = A(n)XB(n) , \quad X(0) = C ,$$

is given by $X = YCZ$. (Here $EY(n) = Y(n + 1)$). An immediate proof follows by substitution.

We now apply this result to finding the third-order linear difference equations whose general solution is $c_1u_1^2 + 2c_2u_1u_2 + c_3u_2^2$ where u_1 and u_2 are linearly independent solutions of

$$\left\{E^2 + p(n)E + q(n)\right\}u = 0 . \tag{12}$$

Without loss of generality, let u_1 and u_2 be determined boundary conditions

$$u_1\big]_{n=0} = 1\,, \qquad EU_1\big]_{n=0} = 0\,,$$
$$u_2\big]_{n=0} = 0\,, \qquad EU_2\big]_{n=0} = 1\,.$$

Setting $Eu = v$, (12) is equivalent to

$$\begin{aligned} Eu &= v\,, \\ Ev &= -pv - qu\,. \end{aligned}$$

If we now let

$$A(n) = \begin{Vmatrix} 0 & 1 \\ -q(n) & -p(n) \end{Vmatrix}\,,$$

The matrix solution of

$$Eu = A(n)U\,, \quad U(0) = I\,,$$

is given by

$$U = \begin{Vmatrix} u_1(n) & u_2(n) \\ Eu_1(n) & Eu_2(n) \end{Vmatrix}$$

and the solution of

$$EV = VA(n)^T, \quad V(0) = I\,,$$

by $V = U^T$, the transpose of U. From our lemma, the solution of

$$EX = AXA^T, \quad X(0) = C\,, \tag{13}$$

is given by $X = UCU^T$. Taking C to be the symmetric matrix

$$C = \begin{Vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{Vmatrix},$$

we see that X is given by

$$X = \begin{Vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{Vmatrix}$$

where

$$x_1 = c_1u_1^2 + 2c_2u_1u_2 + c_3u_2^2 ,$$

$$x_2 = c_1u_1Eu_1 + c_2(u_1Eu_2 + u_2Eu_1) + c_3u_2Eu_2 ,$$

$$x_3 = c_1Eu_1^2 + 2c_2(Eu_1)(Eu_2) + c_3Eu_2^2 .$$

Equation (13) can be written as

$$\begin{Vmatrix} Ex_1 & Ex_2 \\ Ex_2 & Ex_3 \end{Vmatrix} = \begin{Vmatrix} 0 & 1 \\ -q & -p \end{Vmatrix} \cdot \begin{Vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & -q \\ 1 & -p \end{Vmatrix}$$

and which is also equivalent to the system

$$Ex_1 = x_3 ,$$

$$Ex_2 = qx_2 - px_3 ,$$

$$Ex_3 = q^2x_1 + 2pqx_2 + p^2x_3 .$$

Eliminating x_2 and x_3, we obtain the third-order linear difference equation corresponding to (5). Similarly, eliminating x_1 and x_2, we obtain the equation whose general solution is $c_1Eu_1^2 + 2c_2(Eu_1)(Eu_2) + c_3Eu_2^2$; eliminating x_1 and x_3, we obtain the equation whose general solution is

$$c_1u_1Eu_1 + c_2(u_1Eu_2 + u_2Eu_1) + c_3u_3Eu_3 .$$

In stating our lemma, we ignored any discussion of the dimensionality of Y and Z. It is clear that the result is valid if A(n) and Y are r x r matrices, B(n) and z s x s matrices, and C and X r x s matrices.

Using the same technique as before, but with much more computation, we can obtain the linear difference equation of order rs whose solutions are the products of order r and one of order s.

REFERENCES

1. C. Appell, Comptes Rendus, XCI (1880), pp. 211-214.
2. G. N. Watson, Bessell Functions, MacMillan, N. Y., 1948, pp. 145-146.
3. R. Bellman, "On the Linear Differential Equations whose Solutions are the Products of Solutions of Two Given Linear Differential Equations," Boll. Un. Mat. Ital. (3) 12 (1957), pp. 12-15.

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(continued from p. 85.)

4. REMARKS

Generalizing these famous conjectures leads to a multitude of conjectures in the Gaussian Integers. Some such as the infinitude of twin primes appears easier to settle and some such as the quadruples of primes seem less attainable than the real case does.

See p. 80 for a First Quadrant Graph of Gaussian Primes.

5. REFERENCES

1. J. H. Jordan and C. J. Potratz, "Complete Residue Systems in the Gaussian Integers," Mathematics Magazine, Vol. 38, No. 1, pp. 1-12 (1965).

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