ON A CHARACTERIZATION OF THE FIBONACCI SEQUENCE

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For the Fibonacci sequence defined by

$$
\begin{aligned}
&F_1 = 1, \quad F_0 = 0 \\
&F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,
\end{aligned} \tag{1}
$$

it is well known that for all n

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{2}$$

We consider the converse problem, i.e., whether or not (1) can be derived from (2).

It is quite easy to prove by induction that if

$$x_{n-1}x_{n+1} - x_n^2 = (-1)^n$$

and

$$x_1 = x_2 = 1,$$

then

$$x_n = x_{n-1} + x_{n-2} .$$

Suppose, however, that x_1 and x_2 are chosen as arbitrary but fixed integers. In this case it will be shown that we cannot conclude (1) from (2), but we do find some interesting results.

Consider the generalized sequence $\{H_n\}$ defined by

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$$H_n = H_{n-1} + H_{n-2} \tag{3}$$
$$H_0 = H_1 = p, \quad p \text{ and } q \text{ are integers.}$$

Under this definition it can be proved that

$$H_{n-1}H_{n+1} - H_n^2 = (-1)^n(p^2 - qp - q^2), \tag{4}$$

and conversely, given equation (4) then (3) must follow. If $p^2 - pq - q^2 = 1$, then (4) is the same as (2).

Therefore let us consider the integral solutions of an equation of the form

$$y^2 - xy - x^2 \pm 1 = 0$$

First of all it can be shown by induction that the Fibonacci numbers do satisfy this equation. If (2) is to characterize the Fibonacci numbers then we must show that the Fibonacci numbers are the only integral solutions to this equation, and then the sequence $\{H_n\}$, with p, q chosen to satisfy

$$y^2 - xy - x^2 - 1 = 0 \tag{5}$$

would be the sequence $\{F_n\}$. However, given examples such as:

$$y = -1, \quad x = 0; \quad y = -2, \quad x = 3; \quad \text{and} \quad y = -5, \quad x = 8$$

it is seen that (2) and (5) do not characterize the Fibonacci sequence.

The characterizing theorem which can be proved is:

Theorem. If x and y are integers such that $y^2 - xy - x^2 \pm 1 = 0$ and

(1) if x and y are positive, then $x = F_{n-1}$, $y = F_n$ for some n,

(2) if x and y are negative, then $x = -F_{n-1}$, $y = -F_n$ for some n,

(3) if either x or y is negative and the other is positive, then $x = F_{n-1}$, $y = -F_n$ or $x = -F_{n-1}$, $y = +F_n$ for some n.

Proof:

(1) Wasteels proved that if x and y are positive integers such that

$$y^2 - xy - x^2 \pm 1 = 0, \quad (y \geq x)$$

then x and y are consecutive Fibonacci numbers [1].

(2) If x and y are negative, then $-x$ and $-y$ are positive and from the first result we know that $-x = F_{n-1}$, $-y = F_n$ for some n. Therefore,

$$x = -F_{n-1}, \quad y = -F_n$$

for some n.

(3) If either x or y is negative and the other positive then: $y^2 - xy - x^2 \pm 1 = 0$ may be written

$$y^2 = |x||y| - x^2 \pm 1 = 0, \quad |x| \geq |y| \tag{6}$$

Let $|y| > 1$. Then from Eq. (6) we find that $|x| > |y|$ and $|x| < 2|y|$. For if $|x| \geq 2|y|$ then

$$x^2 - |x||y| - y^2 \mp 1 = 0 \tag{7}$$

and

$$|x|(|x| - |y|) - y^2 \mp 1 \geq 2|y||y| - y^2 \mp 1 = y^2 \mp 1 > 0.$$

Thus

$$|x/2| < |y| < |x|$$

Let

$$|x| - |y| = |z|$$

Then

$$0 < |z| < |x/2| < |y|$$

and substituting for $|x|$ in (7)

$$(|z| + |y|)^2 - (|z| + |y|)|y| - y^2 \mp 1 = 0$$

or

$$z^2 + |y|\,|z| - y^2 \pm 1 = 0,$$

so that $|z|$ and $|y|$ satisfy Eq. (6) and $|z|$ is smaller than $|x|$ or $|y|$. If $|z| = 1$ then $|y| = 1$ or 2 so that the theorem is true for $|z|$ and $|y|$ and therefore for $|y|$ and $(|y| + |z|)$ or $|x|$.

If $|z| > 1$ we can repeat the above argument and find z_1 such that

$$|z_1| = |y| - |z|$$

which satisfies Eq. (6) and is less than $|z|$.

If $|z_1| > 1$ we can continue this process until eventually we find a $|z_i|$ such that $|z_i| = 1$. Otherwise we would find an infinite sequence of distinct integers less than x and greater than 1.

If $|z_i| = 1$, then the theorem is true for $|z_i|$ and $|z_{i-1}|$ and also for $|z_{i-1}|$ and $(|z_{i-1}| + |z_i|) = |z_{i-2}|$, and similarly for $|z_1|$ and $(|z_1| + |z_2|) = |z|$ and finally for $|x|$ and $|y|$.

We return to the original problem and consider Eq. (4). If

$$p^2 - pq - q^2 - 1 = 0$$

and p and q are positive, then this identity does indeed characterize the Fibonacci sequence. If, however, p and q are both negative then this identity characterizes the negative of the Fibonacci sequence, and if either p or q is negative while the other is positive then this identity may characterize either the Fibonacci sequence or its negative. There is no way in this case to determine which it will be.

REFERENCE

1. Wasteels, M. J., Mathesis (3), 2, 1902, pp. 60-62.

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