# LAH NUMBERS FOR FIBONACCI AND LUCAS POLYNOMIALS

SELMO TAUBER
Portland State College, Portland, Oregon

## I. INTRODUCTION

In [1] the Fibonacci and Lucas Polynomials are defined as follows:

$$(1) \qquad f_0(x) = 0, \quad f_1(x) = 1, \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \quad n \geq 0,$$

and,

$$(2) \qquad \mathrm{Luc}_0(x) = 2, \quad \mathrm{Luc}_n(x) = f_{n+1}(x) + f_{n-1}(x), \quad n > 0.$$

It is easily seen that

$$(3) \qquad f_n(x) = \sum_{m=0}^{[n/2]} \mathrm{Fi}_n^{n-2m} x^{n-2m-1} = \sum_{s=0}^{n} \mathrm{Fi}_n^s x^{s-1},$$

where n and s have same parity (n - s - 2k), [n/2] is the largest integer contained in n/2, i. e.,

$$(4) \qquad [n/2] = \begin{cases} (n/2) & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and,

$$(5) \qquad \mathrm{Fi}_n^{n-2m} = \binom{n-m-1}{m}, \quad \mathrm{Fi}_n^s = \binom{(n+s-2)/2}{(n-s)/2},$$
$$\mathrm{Fi}_n^s = 0, \quad \text{for } s < 1, \; n < 1, \; n < s, \; n - s = 2k + 1.$$

It follows from (2) that

$$\text{(6)}\qquad \mathrm{Luc}_n(x) = \sum_{m=0}^{[(n+1)/2]} \mathrm{Lu}_n^{n-2m}\, x^{n-2m} = \sum_{s=0}^{n} \mathrm{Lu}_n^s\, x^s ,$$

where $n$ and $s$ have same parity $(n - s = 2k)$,

and,

$$\text{(7)}\qquad \mathrm{Lu}_n^{n-2m} = \binom{n-m}{m} + \binom{n-m-1}{m-1}, \quad \mathrm{Lu}_n^s = n\left(\frac{n+s}{2} - 1\right)! \Big/ \left(\frac{n-s}{2}\right)!\, s!,$$

$\mathrm{Lu}_n^s = 0$, for $n < 0$, $s < 0$, $n < s$, $n - s = 2k + 1$, and $\mathrm{Lu}_0^0 = 2$.

## 2. FIBONACCI AND LUCAS COEFFICIENTS OF THE SECOND KIND

The numbers $\mathrm{Fi}_n^s$ and $\mathrm{Lu}_n^s$ will be called Fibonacci and Lucas coefficients of the first kind. According to [2], [3], and [4] we call the numbers $\mathrm{fi}_n^s$ and $\mathrm{lu}_n^s$, defined hereafter, Fibonacci and Lucas coefficients of the second kind:

$$\text{(8)}\qquad x^n = \sum_{m=0}^{[(n+1)/2]} \mathrm{fi}_{n+1}^{n+1-2m}\, f_{n+1-2m}(x) = \sum_{s=0}^{n+1} \mathrm{fi}_{n+1}^s\, f_s(x) ,$$

where $n + 1 - s = 2k$, $\mathrm{fi}_n^s = 0$, for $n - s = 2k + 1$, $n < 1$, $s < 1$, $n < s$,

$$\text{(9)}\qquad x^n = \sum_{m=0}^{[n/2]} \mathrm{lu}_n^{n-2m}\, \mathrm{Luc}_{n-2m}(x) = \sum_{s=0}^{n} \mathrm{lu}_n^s\, \mathrm{Luc}_s(x) ,$$

where $n - s = 2k$, $\mathrm{lu}_n^s = 0$, for $n - s = 2k + 1$, $n < 0$, $s < 0$, $n < s$.

According to the general theory seen in [2], [3], and [4], the coefficients $\mathrm{Fi}_n^s$, $\mathrm{fi}_n^s$, on the one hand, and the coefficients $\mathrm{Lu}_n^s$, $\mathrm{lu}_n^s$, on the other hand, are quasi-orthogonal, i.e.,

(10)
$$\sum_{k=0}^{(n-m)/2} Fi_n^{n-2k}\, fi_{n-2k}^m = \delta_n^m ,$$

(11)
$$\sum_{k=0}^{(n-m)/2} Lu_n^{n-2k}\, lu_{n-2k}^m = \delta_n^m ,$$

where $\delta_n^m$ is the Kronecker-delta.

## 3. NUMERICAL VALUES AND RECURRENCE RELATIONS

Using (1) and (2) we obtain the following table of values for $Fi_n^m$ and $Lu_n^m$, limited here to $m, n < 11$ :

| n \ m= | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 0 | 1 | | | | | | | | |
| 3 | 1 | 0 | 1 | | | | | | | |
| 4 | 0 | 2 | 0 | 1 | | | | | | |
| 5 | 1 | 0 | 3 | 0 | 1 | | | | | |
| 6 | 0 | 3 | 0 | 4 | 0 | 1 | | | | |
| 7 | 1 | 0 | 6 | 0 | 5 | 0 | 1 | | | |
| 8 | 0 | 4 | 0 | 10 | 0 | 6 | 0 | 1 | | |
| 9 | 1 | 0 | 10 | 0 | 15 | 0 | 7 | 0 | 1 | |
| 10 | 0 | 5 | 0 | 20 | 0 | 21 | 0 | 8 | 0 | 1 |

$Fi_n^m$

It will be observed that the sum of coefficients in one row is equal to the Fibonacci number corresponding to its n, i.e., $f_n(1) = F_n$.

| n \ m= | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | |
| 2 | 2 | 0 | 1 | | | | | | | | |
| 3 | 0 | 3 | 0 | 1 | | | | | | | |
| 4 | 2 | 0 | 4 | 0 | 1 | | | | | | |
| 5 | 0 | 5 | 0 | 5 | 0 | 1 | | | | | |
| 6 | 2 | 0 | 9 | 0 | 6 | 0 | 1 | | | | |
| 7 | 0 | 7 | 0 | 14 | 0 | 7 | 0 | 1 | | | |
| 8 | 2 | 0 | 16 | 0 | 20 | 0 | 8 | 0 | 1 | | |
| 9 | 0 | 9 | 0 | 30 | 0 | 27 | 0 | 9 | 0 | 1 | |
| 10 | 2 | 0 | 25 | 0 | 50 | 0 | 35 | 0 | 10 | 0 | 1 |

$Lu_n^m$

It is easily seen that

(12) $$Fi_n^m = Fi_{n-2}^m + Fi_{n-1}^{m-1} ,$$

which is satisfied by (5), as can be easily checked.

(13) $$Lu_n^m = Lu_{n-2}^m + Lu_{n-1}^{m-1} ,$$

for $n > 1$, $m > 1$, but for $m = n = 1$ we have

$$Lu_1^1 = \frac{1}{2} Lu_0^0 .$$

It is necessary to introduce the function N(n) which is

(13a) $$N(n) = \begin{cases} 1 & \text{if } n \neq 1 \\ 1/2 & \text{if } n = 1 \end{cases}$$

which allows us to write

(13b) $$Lu_n^m = Lu_{n-2}^m + N(n) Lu_{n-1}^{m-1}$$

for any integer m and n.

According to (9) and (12) of [4] it follows that taking $p = 1$, $k = 2$, the $fi_n^m$-coefficients satisfy the relation

$$fi_n^m = fi_{n-1}^{m-1} - f_{n-1}^{m+1} \quad , \tag{14}$$

and the $lu_n^m$-coefficients the relation

$$lu_n^m = \frac{1}{N(m)}\, lu_{n-1}^{m-1} - \frac{1}{N(m+2)}\, lu_{n-1}^{m+1} \quad , \tag{15}$$

but since $m \geq 2$, $N(m + 2) = 1$, thus

$$lu_n^m = lu_{n-1}^{m-1} \Big/ N(m) - lu_{n-1}^{m+1} \quad . \tag{15a}$$

The numerical values of the Fibonacci and Lucas coefficients of the second kind can be obtained either from (10) and (11) or from (14) and (15a). Thus, for $n, m \leq 10$.

$fi_n^m$

| m \ n= | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 0 | 1 | | | | | | | | |
| 3 | -1 | 0 | 1 | | | | | | | |
| 4 | 0 | -2 | 0 | 1 | | | | | | |
| 5 | 2 | 0 | -3 | 0 | 1 | | | | | |
| 6 | 0 | 5 | 0 | -4 | 0 | 1 | | | | |
| 7 | -5 | 0 | 9 | 0 | -5 | 0 | 1 | | | |
| 8 | 0 | -14 | 0 | 14 | 0 | -6 | 0 | 1 | | |
| 9 | 14 | 0 | -28 | 0 | 20 | 0 | -7 | 0 | 1 | |
| 10 | 0 | 42 | 0 | -48 | 0 | 27 | 0 | -8 | 0 | 1 |

$lu_n^m$.

| m \ n= | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1/2 | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | |
| 2 | -1 | 0 | 1 | | | | | | | | |
| 3 | 0 | -3 | 0 | 1 | | | | | | | |
| 4 | 3 | 0 | -4 | 0 | 1 | | | | | | |
| 5 | 0 | 10 | 0 | -5 | 0 | 1 | | | | | |
| 6 | -10 | 0 | 15 | 0 | -6 | 0 | 1 | | | | |
| 7 | 0 | -35 | 0 | 21 | 0 | -7 | 0 | 1 | | | |
| 8 | 35 | 0 | -56 | 0 | 28 | 0 | -8 | 0 | 1 | | |
| 9 | 0 | 126 | 0 | -84 | 0 | 36 | 0 | -9 | 0 | 1 | |
| 10 | -126 | 0 | 210 | 0 | -120 | 0 | 45 | 0 | -10 | 0 | 1 |

It is easily seen that for n and m having same parity, i.e., $n - m = 2k$, the Fibonacci and Lucas coefficients of the second kind are

$$\text{fi}_n^m = (-1)^{(n-m)/2} \binom{n}{(n-m)/2} m/n , \tag{16}$$

and

$$\text{lu}_n^m = (-1)^m \binom{n}{m} N(m+1) , \tag{17}$$

where $N(m + 1)$, according to (13a) equals 1 if $m \neq 0$, and 1/2 if $m = 0$.

## 4. LAH NUMBERS

According to [5] and [6] the Lucas-Fibonacci and the Fibonacci-Lucas Lah numbers are defined by the two relations

$$\text{Luc}_n(x) = \sum_{m=0}^{n+1} \mu_n^m f_m(x) , \tag{18}$$

and

$$\begin{aligned} f_n(x) &= \sum_{k=0}^{[(n-1)/2]} \lambda_n^{n-1-2k} \text{Luc}_{n-1-2k}(x) \\ &= \sum_{m=0}^{n-1} \lambda_n^m \text{Luc}_m(x) , \end{aligned} \tag{19}$$

where n and m are of the same parity, i.e., $n - m = 2p$.

According to the definition of Lucas polynomials given by (2) it follows that

(20) $$\mu_n^m = \begin{cases} 0 & \text{if } m \neq n \pm 1 \\ 1 & \text{if } m = n \pm 1 \end{cases}$$

and

(21) $$\lambda_n^m = (-1)^{(n-m-1)/2} N(m + 1) ,$$

where $n$ and $m$ are of opposite parity, i.e., $n - m = 2k + 1$, and $N(m)$ is defined by (13a).

According to (8) and (9) of [5], and (3a) and (3b) of [6] we obtain

(22) $$\lambda_n^m = \sum_{s=m}^{n} Fi_n^s \, lu_{s-1}^m = (-1)^{(n-m-1)/2} N(m + 1) ,$$

where $n$ and $m$ have different parity, i.e., $n - m = 2k + 1$, and

(23) $$\sum_{s=m-1}^{n} Lu_n^s \, fi_{s+1}^m = \mu_n^m = \begin{cases} 0 & \text{if } m \neq n \pm 1 \\ 1 & \text{if } m = n \pm 1 \end{cases}$$

## REFERENCES


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