# ADVANCED PROBLEMS AND SOLUTIONS 

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-143 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let $\left\{H_{n}\right\}$ be a generalized Fibonacci sequence and, by the recurrence relation, extend the definition to include negative subscripts. Show that

$$
\begin{equation*}
L_{2 j+1} \sum_{k=0}^{n} H_{(2 j+1) k}^{2}=H(2 j+1)(n+1) H(2 j+1) n-H_{0} H_{-(2 j+1)}, \tag{i}
\end{equation*}
$$

(ii)

$$
L_{2 j+1} \sum_{k=0}^{n} H_{(2 j+1) k}=H_{(2 j+1)(n+1)}-H_{-(2 j+1)},
$$

(iii)

$$
\mathrm{L}_{2 \mathrm{j}} \sum_{\mathrm{k}=0}^{\mathrm{n}}(-1)^{\mathrm{k}_{H_{2 j k}^{2}}^{2}=(-1)^{\mathrm{n}} \mathrm{H}_{2 j(n+1)} \mathrm{H}_{2 j n}-\mathrm{H}_{0} \mathrm{H}_{-2 j}, ~, ~, ~}
$$

and derive an expression for
(iv)

$$
\sum_{k=0}^{n}(-1)^{k_{H_{2 j k}}}
$$

H-144 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.
A. Put

$$
[(1-x)(1-y)(1-a x)(1-b y)]^{-1}=\sum_{m, n=0}^{\infty} A_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} A_{n, n} x^{n}=\frac{1-a b x^{2}}{(1-x)(1-a x)(1-b x)(1-a b x)}
$$

B. Put

$$
(1-x)^{-1}(1-y)^{-1}(1-a x y)^{-} \lambda=\sum_{m, n=0}^{\infty} B_{m, n} x^{m} y^{n}
$$

Show that

$$
\sum_{n=0}^{\infty} B_{n, n} x^{n}=(1-x)^{-1}(1-a x)^{-\lambda}
$$

耳-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va. If

$$
n=p_{1}^{e_{1}} p_{2}^{e_{2}} \ldots p_{r}^{e_{r}}
$$

is the canonical factorization of $n$, let $\lambda(n)=e_{1}+\ldots+e_{r}$ Show that $\lambda(n)$ $\leq \lambda\left(F_{n}\right)+1$ for all $n$, where $F_{n}$ is the $n{ }^{\text {th }}$ Fibonacci number.

H-146 Proposed by J. A. H. Hunter, Toronto, Canada.
Let $P_{n}$ be the $n^{\text {th }}$ Pell number defined by $P_{1}=1, P_{2}=2$, and $P_{n+2}$ $=2 P_{n+1}+P_{n}$. Prove that the only square Pell numbers are $P_{1}=1$, and $P_{7}$ $=169$.

H-147 Proposed by George Ledin, Jr., University of San Francisco, San Francisco, California.
Find the following limits. $F_{k}$ is the $k^{\text {th }}$ Fibonacci number, $L_{k}$ is the $\mathrm{k}^{\text {th }}$ Lucas number, $\pi=3.14159 \cdots, \alpha=(1+\sqrt{5}) / 2=1.61803 \cdots, \mathrm{k}^{\mathrm{k}}=1$, $2,3, \cdots$ 。

$$
\begin{aligned}
& X_{1}=\lim _{n \rightarrow \infty} \frac{F_{F_{n+1}}}{F_{F_{n}}^{\alpha}} \\
& X_{2}=\lim _{n \rightarrow 0}\left|\frac{\mathrm{~F}{ }_{\mathrm{n}} \mathrm{~m}}{\mathrm{n}^{m}}\right| \\
& \left.\mathrm{X}_{3}=\lim _{\mathrm{n}}\left|\frac{\mathrm{~F}}{\mathrm{~m}}\right| \frac{\mathrm{n}^{\mathrm{m}}}{\mathrm{~F}_{\mathrm{n}}^{\mathrm{m}}} \right\rvert\, \\
& \mathrm{X}_{4}=\lim _{\mathrm{n} \rightarrow 0}\left|\frac{\mathrm{~F}{ }^{m}{ }_{\mathrm{n}}{ }^{m-1} \mathrm{~F}_{\mathrm{n}}}{}\right| \\
& X_{5}=\lim _{n \rightarrow 0}\left|\frac{L_{n}-2}{n}\right|
\end{aligned}
$$

## SOLUTIONS

SUM DAY

H-103 Proposed by David Zeitlin, Minneapolis, Minnesota.

Show that

$$
8 \sum_{k=0}^{n} F_{3 k+1} F_{3 k+2} F_{6 k+3}=F_{3 n+3}^{4}
$$

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.
We suppose known that

$$
F_{2 X+1}=F_{X}^{2}+F_{X+1}^{2}
$$

and that

$$
F_{x} F_{x+2}=(-1)^{x+1}+F_{x+1}^{2}
$$

Then

$$
\begin{aligned}
\mathrm{F}_{2 \mathrm{x}-3} & =\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-1}^{2}=\left(\mathrm{F}_{\mathrm{x}}-\mathrm{F}_{\mathrm{X}-2}\right)^{2}+\left(\mathrm{F}_{\mathrm{X}-1}-\mathrm{F}_{\mathrm{X}-2}\right)^{2} \\
& =\mathrm{F}_{\mathrm{x}}^{2}+\mathrm{F}_{\mathrm{x}-1}^{2}+\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-3}^{2}-2 \mathrm{~F}_{\mathrm{x}} \mathrm{~F}_{\mathrm{x}-2}-2 \mathrm{~F}_{\mathrm{x}-1} \mathrm{~F}_{\mathrm{x}-3} \\
& =\mathrm{F}_{\mathrm{x}}^{2}+\mathrm{F}_{\mathrm{x}-1}^{2}+\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-3}^{2}-2\left((-1)^{\mathrm{x}-1}+\mathrm{F}_{\mathrm{x}-1}^{2}+(-1)^{\mathrm{x}-2}+\mathrm{F}_{\mathrm{x}-2}^{2}\right) \\
& =\mathrm{F}_{\mathrm{x}}^{2}-\mathrm{F}_{\mathrm{x}-1}^{2}-\mathrm{F}_{\mathrm{x}-2}^{2}+\mathrm{F}_{\mathrm{x}-3}^{2}=\mathrm{F}_{\mathrm{x}}^{2}-\mathrm{F}_{2 \mathrm{x}-3}+\mathrm{F}_{\mathrm{x}-3}^{2},
\end{aligned}
$$

whence

$$
2 \mathrm{~F}_{2 \mathrm{X}-3}=\mathrm{F}_{\mathrm{X}}^{2}+\mathrm{F}_{\mathrm{X}-3}^{2}
$$

Now, the identity to be proved is clearly true for $\mathrm{n}=0$ and we need only show that the right- and left-hand sides increase by the same amount when n is replaced by $n+1$. The right-hand increase is

$$
\mathrm{F}_{3 \mathrm{n}+6}^{4}-\mathrm{F}_{3 \mathrm{n}+3}^{4}=\left(\mathrm{F}_{3 \mathrm{n}+6}-\mathrm{F}_{3 \mathrm{n}+3}\right)\left(\mathrm{F}_{3 \mathrm{n}+6}+\mathrm{F}_{3 \mathrm{n}+3}\right)\left(\mathrm{F}_{3 \mathrm{n}+6}^{2}+\mathrm{F}_{3 \mathrm{n}+3}^{2}\right)
$$

The first factor is

$$
\mathrm{F}_{3 \mathrm{n}+5}+\mathrm{F}_{3 \mathrm{n}+4}-\mathrm{F}_{3 \mathrm{n}+3}=2 \mathrm{~F}_{3 \mathrm{n}+4}
$$

The second is

$$
\mathrm{F}_{3 \mathrm{n}+5}+\mathrm{F}_{3 \mathrm{n}+4}+\mathrm{F}_{3 \mathrm{n}+3}=2 \mathrm{~F}_{3 \mathrm{n}+5}
$$

Thus the total is

$$
4 \mathrm{~F}_{3 n+4} \mathrm{~F}_{3 n+5}\left(\mathrm{~F}_{3 \mathrm{n}+6}^{2}+\mathrm{F}_{3 \mathrm{n}+3}^{2}\right)
$$

The left-hand side increase is

$$
8 \mathrm{~F}_{3 \mathrm{n}+4} \mathrm{~F}_{3 \mathrm{n}+5} \mathrm{~F}_{6 \mathrm{n}+9} .
$$

These increases are equal if

$$
2 \mathrm{~F}_{6 \mathrm{n}+9}=\mathrm{F}_{3 \mathrm{n}+6}^{2}+\mathrm{F}_{3 \mathrm{n}+3}^{2},
$$

which we have already proved.
Also solved by F. D. Parker, Charles R. Wall, and J. Ramanna.

## GENERATOR TROUBLE

H-104 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show

$$
\frac{L_{m} x}{1-5 F_{m} x+(-1)^{m+1} 5 x^{2}}=\sum_{k=0}^{\infty} 5^{k}\left(F_{2 m k}+x L_{(2 k+1) m}\right) x^{2 k}
$$

where $L_{m}$ and $F_{m}$ are the $m^{\text {th }}$ Lucas and Fibonacci numbers, respectively. Solution by David Zeitlin, Minneapolis, Minnesota.

Using (14) and (16) in my paper, "On Summation Formulas for Fibonacci and Lucas Numbers," this Quarterly, Vol. 2, No. 2, 1964, pp. 105-107, we obtain, respectively

$$
\begin{equation*}
\left(1-L_{2 m} y+y^{2}\right) \sum_{k=0}^{\infty} F_{2 m k} y^{k}=F_{2 m y} \tag{1}
\end{equation*}
$$

(2)

$$
\begin{aligned}
\left(1-L_{2 m y}+y^{2}\right) \sum_{k=0}^{\infty} L_{(2 k+1) m} y^{k} & =L_{m}+\left(L_{3 m}-L_{m} L_{2 m}\right) y \\
& = \\
& =L_{m}+(-1)^{m+1} L_{m} y
\end{aligned}
$$

since

$$
L_{m}=(-1)^{m+1}\left(L_{3 m}-L_{m} L_{2 m}\right)
$$

For $y=5 x^{2}$, we obtain

$$
\begin{aligned}
\sum_{k=0}^{\infty} 5^{k}\left(F_{2 m k}\right. & \left.+x L_{(2 k+1) m}\right) x^{2 k} \\
& =\frac{L_{m} x\left(1+5 F_{m} x+5(-1)^{m+1} x^{2}\right)}{1-5 L_{2 m} x^{2}+25 x^{4}} \equiv \frac{L_{m} x}{1-5 F_{m} x+5(-1)^{m+1} x^{2}}
\end{aligned}
$$

since

$$
\mathrm{L}_{2 \mathrm{~m}}=2(-1)^{\mathrm{m}}+5 \mathrm{~F}_{\mathrm{m}}^{2}
$$

and so

$$
1-5 L_{2 m} x^{2}+25 x^{4}=\left(1+5 F_{m} x+5(-1)^{m+1} x^{2}\right)\left(1-5 F_{m}^{\left.x+5(-1)^{m+1} x^{2}\right)}\right.
$$

Also solved by Anthony G. Shannon (Australia).

## OF PRIME INTEREST

H-105 Proposed by Edgar Karst, Norman, Oklahoma, and S. O. Rorem, Davenport, lowa.
Show for all positive integral $n$ and primes $p>2$ that

$$
(\mathrm{n}+1)^{\mathrm{p}}-\mathrm{n}^{\mathrm{p}}=6 \mathrm{~N}+1
$$

where N is a positive integer. Generalize.
Solution by E. W. Bowen, University of New England, Australia.
Let b be a prime, m a positive integer, and $\mu$ the least positive residue of $m$ modulo $b-1$, $i_{0} e_{0}$, for some integer $k, m=k(b-1)+\mu$ where $0<\mu<b-1$.

Clearly $\mathrm{n}^{\mathrm{m}}=0=\mathrm{n}^{\mu}(\bmod b)$ if n is a multiple of b . If n is not a multiple of $b$, we have by Fermat's theorem,

$$
\mathrm{n}^{\mathrm{b}-1} \equiv 1(\bmod \mathrm{~b})
$$

from which we infer

$$
\mathrm{n}^{\mathrm{m}} \equiv \mathrm{n}^{\mathrm{k}(\mathrm{~b}-1)+\mu} \equiv 1_{\mathrm{n}}^{\mathrm{k}} \mathrm{n}^{\mu} \equiv \mathrm{n} \quad(\bmod b)
$$

Thus, for all integers $n$ we have

$$
\mathrm{n}^{\mathrm{m}} \equiv \mathrm{n}^{\mu}(\bmod b)
$$

Using $\Delta$ to denote the difference operator by which

$$
\Delta f(n)=f(n+1)-f(n)
$$

and noting that $\Delta \Delta_{n} \mu=\mu!$, we obtain

$$
\Delta^{\mu} \mathrm{n}^{\mathrm{m}} \equiv \mu \mathrm{~m}!(\bmod \mathrm{b})
$$

and in particular, with $\mu=1$,

$$
\Delta \mathrm{n}^{\mathrm{m}} \equiv 1(\bmod \mathrm{~b}) \text { if } \mathrm{m}=1(\bmod \mathrm{~b}-1)
$$

If $b_{1}, b_{2}, \cdots, b_{S}$ are different primes, we infer immediately that
$\star \Delta n^{m} \equiv 1\left(\bmod b_{1} b_{2} \cdots b_{S}\right)$ if $m \equiv 1\left(\bmod \left(b_{1}-1\right) \cdots\left(b_{S}-1\right)\right)$.

This is a generalization of the required result since, with 2 and 3 as the primes, $b_{i}$, we find that for any odd $m$, and hence for $m=p>2, \Delta n^{m}=1(\bmod$ 6 ), i. e., $(n+1)^{m}-n^{m}=6 N+1$ for some integer $N ; N$ is obviously positive when $n$ is positive and $m>2$.

Examples of other results obtained from $\star$ are:

$$
\begin{aligned}
& \Delta \mathrm{n}^{\mathrm{m}} \equiv 1(\bmod 10) \text { if } \mathrm{m}=4 \mathrm{k}+1 \\
& \Delta \mathrm{n}^{\mathrm{m}} \equiv 1(\bmod 30) \text { if } \mathrm{m}=8 \mathrm{k}+1
\end{aligned}
$$

Summing gives a further generalization of *:

$$
(n+r)^{m}-n^{m} \equiv r \quad\left(\bmod b_{1} \cdots b_{s}\right)
$$

if

$$
m \equiv 1\left(\bmod \left(b_{1}-1\right) \cdots\left(b_{S}-1\right)\right)
$$

Also solved by J. A. H. Hunter, Brother Alfred Brousseau, David Singmaster, Steven Weintraub, and Anthony Shannon.

## BUY MY NOMIAL?

H-106 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.
Show that
(a)

$$
\sum_{k=0}^{n}\binom{n}{k}^{2} L_{2 k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} L_{n-k}
$$

(b)

$$
\sum_{k=0}^{n}\binom{n}{k}^{2} F_{2 k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} F_{n-k}
$$

Solution by David Zeitlin, Minneapolis, Minnesota.
If

$$
P(x) \equiv \sum_{k=0}^{n}\binom{n}{k}^{2} x^{k}
$$

and

$$
Q(x) \equiv \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(x-1)^{n-k},
$$

then $P(x) \equiv Q(x)$ is a known identity (see Elementary Problem E799, American Math. Monthly, 1948, p. 30). If $\alpha$ and $\beta$ are roots of $x^{2}-x-1=0$, then

$$
\mathrm{L}_{\mathrm{n}}=\alpha^{\mathrm{n}}+\beta^{\mathrm{n}}, \quad \mathrm{~F}_{\mathrm{n}}=\left(\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}\right) / \sqrt{5}
$$

and thus
(a)

$$
\mathbf{P}\left(\alpha^{2}\right)+\mathbf{P}\left(\beta^{2}\right)=\mathbf{Q}\left(\alpha^{2}\right)+\mathbf{Q}\left(\beta^{2}\right)
$$

(b)

$$
\frac{\mathrm{P}\left(\alpha^{2}\right)-\mathrm{P}\left(\beta^{2}\right)}{\sqrt{5}}=\frac{\mathrm{Q}\left(\alpha^{2}\right)-\mathrm{Q}\left(\beta^{2}\right)}{\sqrt{5}}
$$

## BE DETERMINANT!

H-107 Proposed by Vladimir Ivanoff, San Carlos, California.
Show that
for all integers $p, q, r$ and $n$.

Solution by C. C. Yalavigi, Government College, Mercara, India.
Let
(1)

$$
D=\left|\begin{array}{lllll}
F_{p_{1}+r n} & \cdots & \cdots & F_{p_{1}+n} & F_{p_{1}} \\
F_{p_{2}+r n} & \cdots & \cdots & F_{p_{2}+n} & F_{p_{2}} \\
\vdots & & & & \\
F_{p_{r+1}+r n} & \cdots & \cdots & F_{p_{r+1}+n} & F F_{p_{r+1}}
\end{array}\right|
$$

On simplifying the first column of this determinant by the use of

$$
F_{i+j}=F_{i+1} F_{j}+F_{i} F_{j-1}
$$

it is easy to show that

$$
D=F_{r n}\left|\begin{array}{lllll}
F_{p_{1}+1} & F_{p_{1}+(r-1) n} & \cdots & F_{p_{1}+n} & F_{p_{1}} \\
F_{p_{2}+1} & F_{p_{2}+(r-1) n} & \cdots & F_{p_{2}+n} & F_{p_{2}} \\
\vdots & & & & \\
F_{p_{r+1}+1} & F_{p_{r+1}+(r-1) n} & \cdots & F_{p_{r+1}+n} & F_{p_{r+1}}
\end{array}\right|
$$

(2)

$$
+F_{r n-1}\left|\begin{array}{lllll}
F_{p_{1}} & F_{p_{1}+(r-1) n} & \cdots & F_{p_{1}+n} & F_{p_{1}} \\
F_{p_{2}} & F_{p_{2}+(r-1) n} & \cdots & F_{p_{2}+n} & F_{p_{2}} \\
\vdots & & & & \\
p_{p_{r+1}} & F_{p_{r+1}+(r-1) n} & \cdots & F_{p_{r+1}+n} & F_{p_{r+1}}
\end{array}\right|
$$

when the subtraction of $F_{n}$ times the first column and $F_{n-1}$ times the last column from the last but one column in the first determinant reduces it to zero and the second determinant also vanishes.

Therefore the desired result follows for $r=2$ 。
Also solved by F. D. Parker, David Zeitlin, Anthony Shannon, C. B. A. Peck, Douglas Lind, William Lombard, Charles R. Wall.

