Edited by RAYMOND E. WHITNEY Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-143 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let  $\{H_n\}$  be a generalized Fibonacci sequence and, by the recurrence relation, extend the definition to include negative subscripts. Show that

(i) 
$$L_{2j+1} \sum_{k=0}^{n} H_{(2j+1)k}^{2} = H_{(2j+1)(n+1)} H_{(2j+1)n} - H_0 H_{-(2j+1)}$$
,

(ii) 
$$L_{2j+1} \sum_{k=0}^{n} H_{(2j+1)k} = H_{(2j+1)(n+1)} - H_{-(2j+1)}$$

(iii) 
$$L_{2j} \sum_{k=0}^{n} (-1)^{k} H_{2jk}^{2} = (-1)^{n} H_{2j(n+1)} H_{2jn} - H_{0} H_{-2j}$$

and derive an expression for

(iv)

$$\sum_{k=0}^{n} (-1)^{k} H_{2jk}$$

H-144 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

A. Put

$$[(1 - x)(1 - y)(1 - ax)(1 - by)]^{-1} = \sum_{m,n=0}^{\infty} A_{m,n} x^m y^n$$
.

Show that

$$\sum_{n=0}^{\infty} A_{n,n} x^{n} = \frac{1 - abx^{2}}{(1 - x)(1 - ax)(1 - bx)(1 - abx)} .$$

B. Put

$$(1 - x)^{-1}(1 - y)^{-1}(1 - axy)^{-} \lambda = \sum_{m,n=0}^{\infty} B_{m,n} x^{m} y^{n}$$
.

Show that

$$\sum_{n=0}^{\infty} B_{n,n} x^{n} = (1 - x)^{-1} (1 - ax)^{-\lambda}.$$

B-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

If

$$\mathbf{n} = \mathbf{p}_1^{\mathbf{e}_1} \mathbf{p}_2^{\mathbf{e}_2} \cdots \mathbf{p}_r^{\mathbf{e}_r}$$

is the canonical factorization of n, let  $\lambda(n) = e_1 + \cdots + e_r$ . Show that  $\lambda(n) \leq \lambda(F_n) + 1$  for all n, where  $F_n$  is the n<sup>th</sup> Fibonacci number.

352

H-146 Proposed by J. A. H. Hunter, Toronto, Canada.

Let  $P_n$  be the n<sup>th</sup> Pell number defined by  $P_1 = 1$ ,  $P_2 = 2$ , and  $P_{n+2} = 2P_{n+1} + P_n$ . Prove that the only square Pell numbers are  $P_1 = 1$ , and  $P_7 = 169$ .

## H–147 Proposed by George Ledin, Jr., University of San Francisco, San Francisco, California.

Find the following limits.  $F_k$  is the k<sup>th</sup> Fibonacci number,  $L_k$  is the k<sup>th</sup> Lucas number,  $\pi = 3.14159 \cdots$ ,  $\alpha = (1 + \sqrt{5})/2 = 1.61803 \cdots$ , m = 1, 2, 3,  $\cdots$ .

$$X_{1} = \lim_{n \to \infty} \frac{F_{F_{n+1}}}{F_{F_{n}}^{\alpha}}$$

$$X_{2} = \lim_{n \to 0} \left| \frac{F_{m}}{n^{m}} \right|$$

$$X_{3} = \lim_{n \to 0} \left| \frac{F_{m}}{F_{n}^{m}} \right|$$

$$X_{4} = \lim_{n \to 0} \left| \frac{F_{m}}{n^{m-1}F_{n}} \right|$$

$$X_{5} = \lim_{n \to 0} \left| \frac{L_{n} - 2}{n} \right|$$

#### SOLUTIONS

### SUM DAY

H-103 Proposed by David Zeitlin, Minneapolis, Minnesota.

### Show that

$$8 \sum_{k=0}^{n} F_{3k+1} F_{3k+2} F_{6k+3} = F_{3n+3}^{4}$$
.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.

We suppose known that

$$F_{2X+1} = F_X^2 + F_{X+1}^2$$

and that

$$F_{X}F_{X+2} = (-1)^{X+1} + F_{X+1}^{2}$$
.

Then

$$\begin{aligned} F_{2X-3} &= F_{X-2}^2 + F_{X-1}^2 = (F_X - F_{X-2})^2 + (F_{X-1} - F_{X-2})^2 \\ &= F_X^2 + F_{X-1}^2 + F_{X-2}^2 + F_{X-3}^2 - 2F_XF_{X-2} - 2F_{X-1}F_{X-3} \\ &= F_X^2 + F_{X-1}^2 + F_{X-2}^2 + F_{X-3}^2 - 2((-1)^{X-1} + F_{X-1}^2 + (-1)^{X-2} + F_{X-2}^2) \\ &= F_X^2 - F_{X-1}^2 - F_{X-2}^2 + F_{X-3}^2 = F_X^2 - F_{2X-3} + F_{X-3}^2 , \end{aligned}$$

whence

$$2F_{2X-3} = F_X^2 + F_{X-3}^2$$
.

Now, the identity to be proved is clearly true for n = 0 and we need only show that the right- and left-hand sides increase by the same amount when n is replaced by n + 1. The right-hand increase is

$$F_{3n+6}^4 - F_{3n+3}^4 = (F_{3n+6} - F_{3n+3})(F_{3n+6} + F_{3n+3})(F_{3n+6}^2 + F_{3n+3}^2)$$
.

The first factor is

$$F_{3n+5} + F_{3n+4} - F_{3n+3} = 2F_{3n+4}$$
.

353

1968]

[Dec.

The second is

$$F_{3n+5} + F_{3n+4} + F_{3n+3} = 2F_{3n+5}$$
.

Thus the total is

$$4F_{3n+4}F_{3n+5}(F_{3n+6}^2 + F_{3n+3}^2)$$
.

The left-hand side increase is

$$8F_{3n+4}F_{3n+5}F_{6n+9}$$

These increases are equal if

$$2F_{6n+9} = F_{3n+6}^2 + F_{3n+3}^2$$
,

which we have already proved.

Also solved by F. D. Parker, Charles R. Wall, and J. Ramanna.

## GENERATOR TROUBLE

H–104 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show

$$\frac{L_{m}x}{1 - 5F_{m}x + (-1)^{m+1}5x^{2}} = \sum_{k=0}^{\infty} 5^{k}(F_{2mk} + xL_{(2k+1)m})x^{2k},$$

where  $L_m$  and  $F_m$  are the m<sup>th</sup> Lucas and Fibonacci numbers, respectively. Solution by David Zeitlin, Minneapolis, Minnesota.

Using (14) and (16) in my paper, "On Summation Formulas for Fibonacci and Lucas Numbers," this Quarterly, Vol. 2, No. 2, 1964, pp. 105-107, we obtain, respectively

(1) 
$$(1 - L_{2m}y + y^2) \sum_{k=0}^{\infty} F_{2mk}y^k = F_{2m}y$$

~~

(2) 
$$(1 - L_{2m}y + y^2) \sum_{k=0}^{\infty} L_{(2k+1)m} y^k = L_m + (L_{3m} - L_m L_{2m}) y$$
  
=  $L_m + (-1)^{m+1} L_m y$ ,

since

1968]

$$L_m = (-1)^{m+1} (L_{3m} - L_m L_{2m})$$
.

For  $y = 5x^2$ , we obtain

$$\sum_{k=0}^{\infty} 5^{k} (F_{2mk} + xL_{(2k+1)m}) x^{2k}$$
$$= \frac{L_{m} x (1 + 5F_{m} x + 5(-1)^{m+1} x^{2})}{1 - 5L_{2m} x^{2} + 25x^{4}} \equiv \frac{L_{m} x}{1 - 5F_{m} x + 5(-1)^{m+1} x^{2}}$$

since

$$L_{2m} = 2(-1)^m + 5F_m^2$$

and so

$$1 - 5L_{2m}x^{2} + 25x^{4} = (1 + 5F_{m}x + 5(-1)^{m+1}x^{2})(1 - 5F_{m}x + 5(-1)^{m+1}x^{2}).$$

 $\mathbf{b}$ 

Also solved by Anthony G. Shannon (Australia).

#### OF PRIME INTEREST

H–105 Proposed by Edgar Karst, Norman, Oklahoma, and S. O. Rorem, Daven– port, Iowa.

Show for all positive integral n and primes p > 2 that

$$(n + 1)^{p} - n^{p} = 6N + 1$$
,

where N is a positive integer. Generalize.

Solution by E. W. Bowen, University of New England, Australia.

Let b be a prime, m a positive integer, and  $\mu$  the least <u>positive</u> residue of m modulo b - 1, i.e., for some integer k, m = k(b - 1) +  $\mu$  where  $0 < \mu < b - 1$ .

Clearly  $n^m = 0 = n^{\mu} \pmod{b}$  if n is a multiple of b. If n is not a multiple of b, we have by Fermat's theorem,

$$n^{D-1} \equiv 1 \pmod{b},$$

from which we infer

$$n^{m} \equiv n^{k(b-1)+\mu} \equiv 1^{k}n^{\mu} \equiv n \pmod{b} .$$

Thus, for all integers n we have

$$n^m \equiv n^{\mu} \pmod{b}$$
.

Using  $\Delta$  to denote the difference operator by which

$$\Delta f(n) = f(n + 1) - f(n) ,$$

and noting that  $\Delta^{\mu} n^{\mu} = \mu$ ; we obtain

$$\Delta^{\mu} n^{m} \equiv \mu i \pmod{b} .$$

and in particular, with  $\mu = 1$ ,

$$\Delta n^m \equiv 1 \pmod{b}$$
 if  $m = 1 \pmod{b-1}$ .

If  $b_1, b_2, \cdots, b_S$  are different primes, we infer immediately that

\* 
$$\Delta n^m \equiv 1 \pmod{b_1 b_2 \cdots b_S}$$
 if  $m \equiv 1 \pmod{(b_1 - 1) \cdots (b_S - 1)}$ .

This is a generalization of the required result since, with 2 and 3 as the primes, bi, we find that for any odd m, and hence for m = p > 2,  $\Delta n^{m} = 1 \pmod{6}$ , i.e.,  $(n + 1)^{m} - n^{m} = 6N + 1$  for some integer N; N is obviously positive when n is positive and m > 2.

Examples of other results obtained from  $\star$  are:

$$\Delta n^{m} \equiv 1 \pmod{10} \text{ if } m = 4k + 1,$$
  
$$\Delta n^{m} \equiv 1 \pmod{30} \text{ if } m = 8k + 1.$$

Summing gives a further generalization of  $\star$ :

$$(n + r)^{m} - n^{m} \equiv r \pmod{b_1 \cdots b_s}$$

if

1968]

$$m \equiv 1 \pmod{(b_1 - 1) \cdots (b_S - 1)}$$

Also solved by J. A. H. Hunter, Brother Alfred Brousseau, David Singmaster, Steven Weintraub, and Anthony Shannon.

#### BUY MY NOMIAL?

H-106 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Show that

(a)  $\sum_{k=0}^{n} {\binom{n}{k}}^2 L_{2k} = \sum_{k=0}^{n} {\binom{n}{k}} {\binom{n+k}{k}} L_{n-k}$ 

(b) 
$$\sum_{k=0}^{n} {\binom{n}{k}}^2 F_{2k} = \sum_{k=0}^{n} {\binom{n}{k}} {\binom{n+k}{k}} F_{n-k}$$

358

Solution by David Zeitlin, Minneapolis, Minnesota.

If

$$P(x) \equiv \sum_{k=0}^{n} {\binom{n}{k}^2 x^k}$$

and

$$Q(x) \equiv \sum_{k=0}^{n} {n \choose k} {n+k \choose k} (x-1)^{n-k},$$

then  $P(x) \equiv Q(x)$  is a known identity (see Elementary Problem E799, <u>American</u> Math. Monthly, 1948, p. 30). If  $\alpha$  and  $\beta$  are roots of  $x^2 - x - 1 = 0$ , then

$$L_n = \alpha^n + \beta^n$$
,  $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ ,

and thus

(a) 
$$P(\alpha^2) + P(\beta^2) = Q(\alpha^2) + Q(\beta^2)$$

(b) 
$$\frac{P(\alpha^2) - P(\beta^2)}{\sqrt{5}} = \frac{Q(\alpha^2) - Q(\beta^2)}{\sqrt{5}}$$

## **BE DETERMINANT!**

H-107 Proposed by Vladimir Ivanoff, San Carlos, California.

Show that

$$\begin{array}{cccc} F_{p+2n} & F_{p+n} & F_{p} \\ F_{q+2n} & F_{q+n} & F_{q} \\ F_{r+2n} & F_{r+n} & F_{r} \\ and n. \end{array} = 0$$

.

for all integers p,q,r

[Dec.

Solution by C. C. Yalavigi, Government College, Mercara, India. Let

(1) 
$$D = \begin{cases} F_{p_1+rn} & \cdots & F_{p_1+n} & F_{p_1} \\ F_{p_2+rn} & \cdots & F_{p_2+n} & F_{p_2} \\ \vdots \\ F_{p_{r+1}+rn} & \cdots & F_{p_{r+1}+n} & F_{p_{r+1}+n} \end{cases}$$

On simplifying the first column of this determinant by the use of

$$\mathbf{F}_{i+j} = \mathbf{F}_{i+1}\mathbf{F}_j + \mathbf{F}_i\mathbf{F}_{j-1}$$

it is easy to show that

(2)  

$$D = F_{rn} \begin{vmatrix} F_{p_{1}+1} & F_{p_{1}+(r-1)n} & \cdots & F_{p_{1}+n} & F_{p_{1}} \\ F_{p_{2}+1} & F_{p_{2}+(r-1)n} & \cdots & F_{p_{2}+n} & F_{p_{2}} \\ \vdots \\ F_{p_{r+1}+1} & F_{p_{r+1}+(r-1)n} & \cdots & F_{p_{r+1}+n} & F_{p_{r+1}} \end{vmatrix}$$

$$+ F_{rn-1} \begin{vmatrix} F_{p_{1}} & F_{p_{1}+(r-1)n} & \cdots & F_{p_{1}+n} & F_{p_{1}} \\ F_{p_{2}} & F_{p_{2}+(r-1)n} & \cdots & F_{p_{2}+n} & F_{p_{2}} \\ \vdots \\ F_{p_{r+1}} & F_{p_{r+1}+(r-1)n} & \cdots & F_{p_{r+1}+n} & F_{p_{r+1}} \end{vmatrix}$$

when the subtraction of  $F_n$  times the first column and  $F_{n-1}$  times the last column from the last but one column in the first determinant reduces it to zero and the second determinant also vanishes.

Therefore the desired result follows for r = 2.

Also solved by F. D. Parker, David Zeitlin, Anthony Shannon, C. B. A. Peck, Douglas Lind, William Lombard, Charles R. Wall.

古古黄白

1968]