LINEAR RECURSION RELATIONS
LESSON TWO

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Recursion relations can be set up at will. There is, however, a particular type known as the linear recursion relation which by its simplicity, range of application, and interesting mathematical properties deserves special consideration. In this lesson, the linear recursion relation will be described and the method of expressing its terms by means of the roots of an auxiliary equation analyzed. These basic ideas will be applied and amplified in greater detail in succeeding articles.

The term "linear" in mathematics is used by way of analogy with the equation of a straight line in the plane where the variables x and y do not enter in a degree higher than the first. By extension, there are linear equations in more variables which characterize the plane in three-space, the hyperplane in four-space, etc. By further analogy, one speaks of linear differential equations in which the dependent variable and its derivatives are not found in a degree higher than one. In this context it is natural to call a recursion relation of the form:

$$(1) \qquad T_{n+1} = a_1T_n + a_2T_{n-1} + a_3T_{n-2} + \cdots + a_rT_{n-r+1}$$

where the a_i are constants, a linear recursion relation. If a_r is the last non-zero coefficient, then this would be spoken of as a linear recursion relation of order r.

Note that there is no allowance for a constant term. This, however, is no restriction. If, for example,

$$T_{n+1} = 3T_n - 2T_{n-1} + 4T_{n-2} + 8$$

then since

$$T_n = 3T_{n-1} - 2T_{n-2} + 4T_{n-3} + 8$$

it follows by subtraction that

$$T_{n+1} = 4T_n - 5T_{n-1} + 6T_{n-2} - 4T_{n-3} \quad ,$$

so that a linear recursion relation of the standard form (1) can be obtained from this variant.

LINEAR RECURSION RELATION OF THE FIRST ORDER

The linear recursion relation of the first order is

$$(2) \qquad T_{n+1} = r\,T_n \quad ,$$

in which each term is a fixed multiple of the previous term. Evidently, this is the recursion relation of a geometric progression. In terms of the technique that is being developed for relating the terms of the sequence with the roots of an auxiliary equation, we set up the equation corresponding to this recursion relation, namely:

$$(3) \qquad x - r = 0 \quad ,$$

which has the one root r. The term of the sequence can be written as a multiple of the n^{th} power of this root, thus:

$$T_n = (a/r)\,r^n \quad .$$

That this term satisfies the recursion relation (2) follows from (3), since on substituting r for x, we have:

$$r = r \quad ,$$

and on multiplying both sides by r^{n-1},

$$r^n = r \cdot r^{n-1} \, .$$

Note that the powers of the root have the same recursion relation as the terms (2)! So if

$$T_{n+1} = (a/r)\, r^{n+1}$$

and

$$r^{n+1} = r \cdot r^{n} ,$$

$$T_{n+1} = r(a/r)\, r^{n} = r\, T_{n} .$$

Perhaps due to the simplicity of this case, the considerations are confusing! But let us pass on to a second-order linear relation where the operations are not so obvious.

SECOND-ORDER LINEAR RECURSION RELATIONS

In a subsequent article, these relations will be taken up in all possible detail to cover the various situations that may arise. But here we shall start with a simple example to show how the method operates.

Consider then a linear recursion relation

$$T_{n+1} = 5\,T_{n} - 6\,T_{n-1} . \tag{4}$$

If all terms are brought to one side and equated to zero, the result is:

$$T_{n+1} - 5\,T_{n} + 6\,T_{n-1} = 0 . \tag{5}$$

If now the successive terms are replaced by powers of x one obtains the auxiliary equation

$$x^2 - 5x + 6 = 0 \tag{6}$$

whose roots are $r = 3$, $s = 2$. Since they satisfy the equation (5), it follows that

$$r^2 = 5r - 6$$
$$s^2 = 5s - 6$$

Since we may multiply by any power of r or s,

$$r^{n+1} = 5r^n - 6r^{n-1} \qquad (7)$$
$$s^{n+1} = 5s^n - 6s^{n-1}$$

Note that the powers of r and s satisfy the same recursion relation (4) as the terms of the sequence T_n. Hence if we express these terms as linear combinations of powers of r and s, we should obtain expressions that satisfy the recursion relation (4). Set

$$T_{n-1} = ar^{n-1} + bs^{n-1} \qquad (8)$$
$$T_n = ar^n + bs^n$$

where a and b are constants. Then

$$T_{n+1} = 5T_n - 6T_{n-1} = a(5r^n - 6r^{n-1}) + b(5s^n - 6s^{n-1})$$

or

$$T_{n+1} = ar^{n+1} + bs^{n+1}$$

so that the form of the term persists for all values of n once it is established for two initial values.

What this implies is that given any two starting values $T_1 = p$, $T_2 = q$ it is possible to find a sequence

$$T_n = a3^n + b2^n$$

satisfying the recursion relation (4). Consider the particular case $p = 2$, $q = 7$. Then we should have:

$$2 = a \cdot 2 + b \cdot 3$$
$$7 = a \cdot 2^2 + b \cdot 3^2$$

Solving for a and b we obtain $a = -1/2$, $b = 1$, so that in general,

$$T_n = (-1/2)2^n + 3^n .$$

If the roots r and s are real and distinct with $rs \neq 0$, it will always be possible to solve the above set of equations for the determinant of the coefficients of the equations:

$$p = ar + bs$$
$$a = ar^2 + bs^2$$

is

$$\begin{vmatrix} r & s \\ r^2 & s^2 \end{vmatrix} = rs(s - r)$$

which is not zero if $rs \neq 0$ and $s \neq r$.

These considerations can be extended to relations of higher order. For example, suppose we wish to express the terms of a sequence beginning with 3, 8, 14 in the form:

$$T_n = a2^n + b3^n + c5^n .$$

It is simply necessary to set up a recursion relation with roots 2, 3, and 5. Thus the auxiliary equation would be

$$(x - 2)(x - 3)(x - 5) = 0$$

or

$$x^3 = 10x^2 - 31x + 30$$

so that

$$T_{n+1} = 10T_n - 31T_{n-1} + 30T_{n-2}$$

giving sequence terms as follows:

$$3, 8, 14, -18, -374, -2762, -16566, \cdots$$

To express T_n in terms of the powers of the roots use the initial values to form equations as follows.

$$3 = 2a + 3b + 5c$$

$$8 = 4a + 9b + 25c$$

$$14 = 8a + 27b + 125c$$

from which $a = -5/6$, $b = 2$, $c = -4/15$. Thus

$$T_n = (-5/6)2^n + 2 \cdot 3^n + (-4/15)5^n .$$

Evidently, there are many questions that require further study; the case of equal roots of the auxiliary equation; what happens if the roots are irrational; the situation in which the roots are complex; and various combinations of these cases. Such matters will receive attention in a number of subsequent lessons.

PROBLEMS

1. Find the recursion relation for the sequence beginning 3, 10 with terms in the form

$$T_n = a + 2^n b ,$$

and calculate the first ten terms of the sequence.

2. Given the sequence beginning with 5, 12 having a recursion relation

$$T_{n+1} = 8T_n - 15T_{n-1} ,$$

express T_n as a linear combination of powers of the roots of the auxiliary equation.

3. The sequence

$$5,\ 13,\ 61,\ 349,\ 2077,\ 12445,\ 74653,\ 447901, \cdots$$

obeys a linear recursion relation of the second order. Find this relation and express T_n as a linear combination of powers of the roots of the auxiliary equation.

4. A sequence with initial terms 3, 7, 13 has an auxiliary equation

$$x^3 - 6x^2 + 11x - 6 = 0 .$$

Express the term T_n as a linear combination of powers of the roots of this equation.

5. A third-order recursion relation governs the terms of the sequence: 1, 6, 14, 45, 131, 396, 1184, 3555, 10661, 31986, 95954, 287865, 863589 . Determine the coefficients in this recursion relation and express the term T_n as a linear combination of powers of the roots of the auxiliary equation.

LESSON TWO SOLUTIONS

1. $$T_n = -4 + (7/2)\,2^n$$

First ten terms: 3, 10, 24, 52, 108, 220, 444, 892, 1788, 3580.

2. $$T_n = (13/6)\,3^n + (-3/10)\,5^n$$

3. $$T_n = 17/5 + (4/15)\,6^n$$

$$T_{n+1} = 7T_n - 6T_{n-1}$$

4. $$T_n = -2 + 3\cdot 2^n + (-1/3)\,3^n$$

5. $$T_{n+1} = 3T_n + T_{n-1} - 3T_{n-2}$$

$$T_n = 1/4 + (7/8)(-1)^n + (13/24)\,3^n .$$

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