ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-148 Proposed by David Englund, Rockford College, Rockford, Illinois, and Malcolm Tallman, Brooklyn, New York.

Let F_n and L_n denote the Fibonacci and Lucas numbers and show that

$$F_{(2^t n)} = F_n L_n L_{2n} L_{4n} \cdots L_{(2^{t-1} n)} .$$

B-149 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that

$$L_{n+1} L_{n+3} + 4(-1)^{n+1} = 5F_n F_{n+4} .$$

B-150 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that

$$L_n^2 - F_n^2 = 4F_{n-1} F_{n+1} .$$

B-151 Proposed by Hal Leonard, San Jose State College, San Jose, Calif.

Let $m = L_1 + L_2 + \cdots + L_n$ be the sum of the first n Lucas numbers. Let

$$P_n(x) = \prod_{i=1}^{n} (1 + x^{L_i}) = a_0 + a_1 x + \cdots + a_m x^m .$$

Let q_n be the number of integers k such that both $0 < k < m$ and $a_k = 0$. Find a recurrence relation for the q_n.

B-152 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Prove that

$$F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1} .$$

B-153 Proposed by Klaus-Gunther Recke, Gottingen, Germany.

Prove that

$$F_1F_3 + F_2F_6 + F_3F_9 + \cdots + F_nF_{3n} = F_nF_{n+1}F_{2n+1} .$$

SOLUTIONS

GOLDEN RATIO AGAIN?

B-130a Proposed by Sidney Kravitz, Dover, N. Jersey.

An enterprising entrepreneur in an amusement part challenges the public to play the following game. The player is given five equal circular discs which he must drop from a height of one inch onto a larger circle in such a way that the five smaller discs completely cover the larger one. What is the maximum ratio of the diameter of the larger circle to that of the smaller ones so that the player has the possibility of winning?

Partial Solution by the Proposer.

With the centers of the smaller circles placed at the vertices of a regular pentagon, the smaller circles cover the larger one with a ratio of diameters equal to the golden ratio $(1 + \sqrt{5})/2$. There may exist another arrangement of the five circles which results in a smaller ratio.

EVEN AND ODD SEQUENCES

B-131a Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let $\{H_n\}$ be a generalized Fibonacci sequence, i.e., $H_0 = q$, $H_1 = p$, $H_{n+2} = H_{n+1} + H_n$. Extend, by the recursion formula, the definition to include negative subscripts. Show that if $|H_{-n}| = |H_n|$ for all n, then $\{H_n\}$ is a constant multiple of either the Fibonacci or the Lucas sequence.

Solution by David Zeitlin, Minneapolis, Minnesota.

$$H_n = qF_{n+1} + (p - q)F_n ,$$

and since $F_{-n} = (-1)^{n+1}F_n$, we have

$$\left|H_{-n}\right| = \left|(-1)^n(qF_{n-1} - (p - q)F_n)\right| = \left|qF_{n-1} - (p - q)F_n\right| .$$

If $\left|H_1\right| = \left|H_{-1}\right|$, then (a) $p - q = p$ or (b) $p - q = -p$. If (a) holds, then $q = 0$ and $H_n \equiv pF_n$; if (b) holds, then $q = 2p$, and

$$H_n \equiv 2pF_{n+1} - pF_n = pL_n .$$

Remark. Let U_m and V_n be solutions of

$$W_{n+2} = aW_{n+1} + bW_n ,$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = a$ (if $a = b = 1$, then $U_n \equiv F_n$ and $V_n \equiv L_n$). If

$$\left|b^n W_{-n}\right| = \left|W_n\right|$$

for all n, then $\{W_n\}$ is a constant multiple of either $\{U_n\}$ or $\{V_n\}$.

Also solved by Herta T. Freitag, John Ivie, D. V. Jaiswal (India), Bruce W. King, C. B. A. Peck, A. C. Shannon (Australia), and the proposer.

EXPONENT PROBLEM

B-132 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let u and v be relatively prime integers. We say that u belongs to the exponent d modulo v if d is the smallest positive integer such that $u^d \equiv 1 \pmod v$. For $n \geq 3$ show that the exponent to which F_n belongs modulo F_{n+1} is 2 if n is odd and 4 if n is even.

Solution by the proposer.

From

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

we have

$$F_n^2 \equiv (-1)^{n+1} \pmod{F_{n+1}} .$$

Now $F_n \not\equiv 1 \pmod{F_{n+1}}$ as $1 \neq F_n < F_{n+1}$ for $n \geq 3$. If n is odd then $F_n^2 \equiv 1 \pmod{F_{n+1}}$. If n is even then $F_n^2 \equiv -1 \pmod{F_{n+1}}$. Now

$$F_n^3 \equiv -F_n \equiv F_{n-1} \pmod{F_{n+1}}$$

and $F_{n-1} \neq 1$ as $n \geq 4$ (since n is even). But then

$$F_n^4 \equiv (-1)^2 = 1 \pmod{F_{n+1}} .$$

Also solved by D. V. Jaiswal (India) and A. C. Shannon (Australia).

AN OLD PROBLEM IN FIBONACCI CLOTHES

B-133 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let $r = F_{1000}$ and $s = F_{1001}$. Of the two numbers r^s and s^r, which is the larger?

Solution by Phil Mana, University of New Mexico, Albuquerque, N. Mexico.

Since $(\ln x)/x$ is monotonically decreasing for $x > e$,

$$(\ln r)/r > (\ln s)/s$$

or

$$\ln r^{1/r} > \ln s^{1/s} .$$

Since $\ln x$ is monotonically increasing for $x > 0$, this implies that $r^{1/r} > s^{1/s}$. Hence $r^s > s^r$.

Also solved by William D. Jackson, George F. Lowerre, Arthur Marshall, C.B.A. Peck, D. Zeitlin, and the proposer.

A TELESCOPING SUM

B-134 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Define the sequence $\{a_n\}$ by

$$a_1 = a_2 = 1, \quad a_{2k+1} = a_{2k} + a_{2k-1} ,$$

and

$$a_{2k} = a_k$$

for $k \geq 1$. Show that

$$\sum_{k=1}^{n} a_k = a_{2n+1} - 1, \quad \sum_{k=1}^{n} a_{2k-1} = a_{4n+1} - a_{2n+1} .$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{2k} = (a_3 - a_1) + (a_5 - a_3) + \cdots + (a_{2n+1} - a_{2n-1})$$

$$= a_{2n+1} - a_1 = a_{2n+1} - 1 .$$

Then,

$$\sum_{k=1}^{n} a_{2k-1} = \sum_{k=1}^{n} a_{2k-1} + \sum_{k=1}^{n} a_{2k} - \sum_{k=1}^{n} a_{2k} = \sum_{k=1}^{2n} a_k - \sum_{k=1}^{n} a_k$$

$$= (a_{4n+1} - 1) - (a_{2n+1} - 1) = a_{4n+1} - a_{2n+1} .$$

Also solved by L. Carlitz, Herta T. Freitag, John Ivie, D. V. Jaiswal (India), Bruce W. King, George F. Lowerre, C. B. A. Peck, A. C. Shannon (Australia), C. R. Wall, Howard L. Walton, David Zeitlin and the proposer.

GENERALIZED SUMS

B-135 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Put

$$F'_n = \sum_{k=0}^{n-1} F_k 2^{n-k-1}, \quad L'_n = \sum_{k=0}^{n-1} L_k 2^{n-k-1} .$$

Show that, for $n \geq 1$,

$$F'_n = 2^n - F_{n+2}, \quad L'_n = 3 \cdot 2^n - L_{n+2} .$$

Solution by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

Let $\{H_n\}$ be a generalized Fibonacci sequence, and define

$$H'_n = \sum_{k=0}^{n-1} H_k 2^{n-k-1} .$$

Then we claim that

$$\text{(A)} \qquad H'_n = 2^n H_2 - H_{n+2}$$

for all $n \geq 1$.

Identity (A) can be verified for small n; assume that (A) holds for n. Then since

$$2H_{n+2} - H_n = (H_{n+3} - H_{n+1}) + H_{n+2} - (H_{n+2} - H_{n+1}) = H_{n+3}$$

we have

$$H'_{n+1} = \sum_{k=0}^{n} H_k 2^{n-k} = H_n + 2H'_n = 2^{n+1}H_2 - 2H_{n+2} + H_n = 2^{n+1}H_2 - H_{n+3} .$$

Thus (A) holds for all $n \geq 1$. To obtain the identities given by Carlitz, we note that $F_2 = 1$, $L_2 = 3$.

Also solved by Herta T. Freitag, D. V. Jaiswal (India), Bruce W. King, C.B.A. Peck, A. C. Shannon (Australia), David Zeitlin, and the proposer.

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ERRATA

Please make the following correction in the October Elementary Problems and Solutions: In the third equation from the bottom, on p. 292, delete

$$\frac{F_{2k}}{F_{2k+2}} < \frac{F_{2k}}{F_{2k+1}} < \frac{F_{2k+1}}{F_{2k}} < \frac{F_{2k-1}}{F_{2k}}$$

and add, instead,

$$\frac{F_{2k}}{F_{2k+2}} < \frac{F_{2k+2}}{F_{2k+3}} < \frac{F_{2k+1}}{F_{2k+2}} < \frac{F_{2k-1}}{F_{2k}}$$

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[Continued from p. 334.]

Hence, by (13), $p \nmid D'_{2n}$

In each case we have found a reduced arithmetic progression no prime member of which is a factor of a certain D'_{2n}. Hence, by Lemma 1, II), there is an infinitude of composite D'_{2n+1}.

REFERENCES

1. R. D. Carmichael, "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$," Annals of Mathematics, 15 (1913-1914), pp. 30-70.
2. W. J. LeVeque, Topics in Number Theory, I (1958).

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