PSEUDO-FIBONACCI NUMBERS

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Consider the two interlocking recursion formulas

(1) $$O_{i+1} = O_i + P_i$$

(2) $$P_{i+1} = O_{i+1} + \lambda O_i \qquad O_1 = P_1 = 1$$

in which λ is a positive integer.

For reasons which will shortly become apparent we call O_i and P_i pseudo-Fibonacci and pseudo-Lucas numbers, respectively.

In fact, eliminating first the O's and then the P's, from (1) and (2) we get

(3) $$O_{i+1} = 2O_{i+1} + \lambda O_i \qquad O_0 = 0, \quad O_1 = 1$$

(4) $$P_{i+2} = 2P_{i+1} + \lambda P_i \qquad P_0 = 1, \quad P_1 = 1\,.$$

Thus the two numbers defined by (1) and (2) satisfy the same recursion formula but with different initial values.

A Binet-type formula for each of O_n and P_n may be derived from first principles [1]. We leave this as an exercise for the reader.

We shall prove by induction that

(5) $$O_n = \frac{(1 + \sqrt{1+\lambda})^n - (1 - \sqrt{1+\lambda})^n}{2\sqrt{1+\lambda}}$$

(6) $$P_n = \frac{(1 + \sqrt{1+\lambda})^n + (1 - \sqrt{1+\lambda})^n}{2}$$

We introduce the notation $A = 1 + \sqrt{1+\lambda}$ and $B = 1 - \sqrt{1+\lambda}$. From thus it follows that (Received 1963--revised Feb. 1968)

$$A + B = 2 \qquad A - B = 2\sqrt{1+\lambda}$$
$$A^2 = 2A + \lambda \qquad B^2 = 2B + \lambda$$
$$A^2 + B^2 = 2(2+\lambda) \qquad A^2 - B^2 = 4\sqrt{1+\lambda}$$
$$AB = -\lambda$$

Hence

$$O_n = \frac{A^n - B^n}{A - B} \qquad P_n = \frac{A^n + B^n}{2}$$

It is immediately apparent from these two forms of O_n and P_n that $O_1 = P_1 = 1$. Since $O_2 = 2$, (1) is satisfied for $i = 1$. Assume that it is true for $i = 2, 3, \cdots, k$. Then

$$\begin{aligned} O_k + P_k &= \frac{A^k - B^k}{A - B} + \frac{A^k + B^k}{2} \\ &= \frac{2A^k - 2B^k + A^{k+1} - B^{k+1} - A^kB + AB^k}{2(A - B)} \\ &= \frac{A^k(2 - B) - B^k(2 - A) + A^{k+1} - B^{k+1}}{2(A + B)} \\ &= \frac{A^{k+1} - B^{k+1} + A^{k+1} - B^{k+1}}{2(A - B)} \\ &= \frac{A^{k+1} - B^{-k+1}}{A - B} = O_{k+1} \quad . \end{aligned}$$

This completes the proof of (5). A similar proof holds for (6).

If we let $n = -k$ where k is a positive integer we find from (5) and (6) that

$$O_{-k} = -\frac{O_k}{(-\lambda)^k} \qquad \text{and} \qquad P_{-k} = \frac{P_k}{(-\lambda)^k} \quad .$$

It is left as an exercise to show that O_{-k} and P_{-k} satisfy (1) and (2).

In (5) and (6) let $\lambda = 4$. We get

$$(7) \qquad O_n = 2^{n-1}F_n \quad \text{and} \quad P_n = 2^{n-1}L_n \qquad (\lambda = 4)$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively.

Thus identities among O_n and P_n may be transformed by means of the equations in (7) into identities involving Fibonacci and Lucas numbers. Some of the latter will be familiar. The purpose of this article is to find some new or unfamiliar identities among the latter numbers.

We begin with (1) which we write in the form

$$P_i = O_{i+1} - O_i .$$

Let $i = 1, 2, 3, \cdots n$ in this equation. Adding the resulting equations we get

$$(8) \qquad \sum_{i=1}^{n} P_i = O_{n+1} - 1$$

Applying the equations in (7) to (8) we have

$$(8') \qquad \sum_{i=1}^{n} 2^i L_i = 2(2^n F_{n+1} - 1)$$

Next, eliminating O_{i+1} from (1) and (2) yields

$$O_i - \frac{1}{1+\lambda}(P_{i+1} - P_i)$$

Following the procedure used above we get

$$(9) \qquad \sum_{i=1}^{n} O_i = \frac{1}{1+\lambda}(P_{n+1} - 1)$$

(9')
$$\sum_{i=1}^{n} 2^i F_i = \frac{2}{5}(2^n L_{n+1} - 1)$$

A formula for the sum of the first n O's with even numbered subscripts is now derived.

$$\sum_{i=1}^{n} O_{2i} = \sum_{i=1}^{n} \frac{A^{2i} - B^{2i}}{A - B}$$

$$= \frac{1}{A - B}\left[\sum_{i=1}^{n} A^{2i} - \sum_{i=1}^{n} B^{2i}\right]$$

$$= \frac{1}{A - B}\left[\frac{A^2(A^{2n} - 1)}{A^2 - 1} - \frac{B^2(B^{2n} - 1)}{B^2 - 1}\right]$$

(10)

$$= \frac{1}{A - B}\left[\frac{(A^2B^2 - A^2)(A^{2n} - 1) - (A^2B^2 - B^2)(B^{2n} - 1)}{A^2B^2 - A^2 - B^2 + 1}\right]$$

$$= \frac{1}{A - B}\left[\frac{\lambda^2(A^{2n} - B^{2n}) - (A^{2n+2} - B^{2n+2}) + A^2 - B^2)}{\lambda^2 - (A^2 + B^2) + 1}\right]$$

$$= \frac{1}{A - B}\left[\frac{\lambda^2(A - B)O_{2n} - (A - B)O_{2n+2} + (A - B)O_2}{\lambda^2 - 2P_2 + 1}\right]$$

$$= \frac{\lambda^2 O_{2n} - O_{2n+2} + 2}{(\lambda + 1)(\lambda - 3)} \qquad (\lambda \neq 3)$$

Applying recursion formulas (1) and (3) to (10) takes the form

(10a)
$$\sum_{i=1}^{n} O_{2i} = \frac{(\lambda + 1)(\lambda - 4)O_{2n-1} + (\lambda^2 - \lambda - 4)P_{2n-1} + 2}{(\lambda + 1)(\lambda + 3)} \qquad (\lambda \neq 3)$$

From (10a) we get

(10')
$$\sum_{i=1}^{n} 2^{2i} F_{2i} = \frac{4}{5}\,(2^{2n} L_{2n-1} + 1)\ .$$

For the special case in which $\lambda = 3$ we have

$$O_i = \frac{1}{4}\left[3^i - (-1)^i\right]$$

Hence

(10s)
$$\sum_{i=1}^{n} O_{2i} = \frac{1}{4}\left[\sum_{i=1}^{n} 3^i - \sum_{i=1}^{n} (-1)^n\right]$$
$$= \frac{9}{32}\,(9^n - 1) - \frac{n}{4}\ .$$

The following four identities are given without proof. Their derivation follows the same procedure that was used above.

(11)
$$\sum_{i=1}^{n} P_{2i} = \frac{P_{2n+2} - 2\,O_{2n+2} + 2 - \lambda}{\lambda - 3} \qquad (\lambda \neq 3)$$

(11a)
$$\sum_{i=1}^{n} P_{2i} = \frac{(\lambda - 4)\,O_{2n+1} + \lambda\,(\lambda - 2)O_{2n-1} - \lambda + 2}{\lambda - 3} \qquad (\lambda \neq 3)$$

(11s)
$$\sum_{i=1}^{n} P_{2i} = \frac{9}{16}\,(9^n - 1) + \frac{n}{2} \qquad (\lambda = 3)$$

(11')
$$\sum_{i=1}^{n} 2^{2i} L_{2i} = 4(2^{2n} F_{2n} - 1)$$

To find the sum of the first n O's with odd numbered subscripts we use

$$\sum_{i=1}^{n} O_{2i-1} = \sum_{i=1}^{2n} O_i - \sum_{i=1}^{n} O_{2i}$$

and make use of results already obtained. In this manner we get the following four identities:

$$(12) \qquad \sum_{i=1}^{n} O_{2i-1} = \frac{(\lambda - 1)O_{2n+1} - 2\, O_{2n} + 1 - \lambda}{(\lambda + 1)(\lambda - 3)} \qquad (\lambda \neq 3)$$

$$(12a) \qquad \sum_{i=1}^{n} O_{2i-1} = \frac{(\lambda - 4)P_{2n-1} + \lambda(\lambda - 2)P_{2n-2} + 1 - \lambda}{(\lambda + 1)(\lambda - 3)} \qquad (\lambda \neq 3)$$

$$(12s) \qquad \sum_{i=1}^{n} O_{2i-1} = \frac{3}{32}(9^n - 1) + \frac{n}{4} \qquad (\lambda = 3)$$

$$(12') \qquad \sum_{i=1}^{n} 2^{2i-1}F_{2i-1} = \frac{2}{5}\left[2^{2n}L_{2n-2} - 3\right] .$$

The next four are derived in a similar manner.

$$(13) \qquad \sum_{i=1}^{n} P_{2i-1} = \frac{O_{2n+2} - 3O_{2n+1} + 1}{\lambda - 3} \qquad (\lambda \neq 3)$$

$$(13a) \qquad \sum_{i=1}^{n} P_{2i-1} = \frac{(\lambda - 4)O_{2n-1} + \lambda(\lambda - 2)O_{2n-2} + 1}{\lambda - 3} \qquad (\lambda \neq 3)$$

(13s) $$\sum_{i=1}^{n} P_{2i-1} = \frac{3}{16}(9^n - 1) - \frac{n}{2} \qquad (\lambda = 3)$$

(13') $$\sum_{i=1}^{n} 2^{2i-1}L_{2i-1} = 2(2^{2n}F_{2n-2} + 1)$$

We now derive the sum of a series with alternating positive and negative signs.

From (10') and (12') we get

$$\sum_{i=1}^{2n-1} (-1)^{i+1}2^iF_i = \sum_{i=1}^{n} 2^{2i-1}F_{2i-1} - \sum_{i=1}^{n} 2^{2i}F_{2i}$$

$$= \frac{2^{2n+1}}{5}(L_{2n-2} - 2L_{2n-1}) - 2$$

Hence

$$\sum_{i=1}^{2n+1} (-1)^{i+1}2^iF_i = \sum_{i=1}^{n+1} 2^{2i-1}F_{2i-1} - \sum_{i=1}^{n} 2^{2i}F_{2i}$$

$$= \frac{2^{2n+2}}{5}(2L_{2n} - L_{2n-1}) - 2 .$$

From the last two equations we conclude that

(14) $$\sum_{i=1}^{n} (-1)^{i+1}2^iF_i = \frac{(-2)^{n+1}}{5}(2L_n - L_{n-1}) - 2 .$$

In like manner, beginning with (11') and (13') we get

$$\sum_{i=1}^{n} (-1)^{i+1} 2^i L_i = (-2)^{n+1}(F_{n-2} - 2F_{n-1}) + 6 \quad . \tag{15}$$

The following identities involve sums of squares. Derivation is given for the first one only.

In several cases the final term is

$$\pm \frac{\lambda}{1+\lambda}\left[(1 - (-\lambda)^n)\right] .$$

For brevity we shall denote this by $\pm R$.

$$\sum_{i=1}^{n} O_i^2 = \frac{1}{2(1+\lambda)}\left[\frac{(\lambda - 4)O_{2n} + \lambda(\lambda - 2)O_{2n-1} + 2 - \lambda}{\lambda - 3} + R\right] \quad (\lambda \neq 3) \tag{16}$$

$$\sum_{i=1}^{n} O_i^2 = \frac{1}{128}\left[9^{n+1} - 8n + 4(-3)^{n+1} + 3\right] \quad (\lambda = 3) \tag{16s}$$

$$\sum_{i=1}^{n} 2^{2i} F_i^2 = \frac{2}{5}\left[2^{2n+1} F_{2n-1} + \frac{(-4)^{n+1} - 6}{5}\right] \tag{16'}$$

$$\sum_{i=1}^{n} P_i^2 = \frac{1}{2}\left[\frac{(\lambda - 4)O_{2n} + (\lambda - 2)O_{2n-1}}{\lambda - 3} - R\right] \quad (\lambda \neq 3) \tag{17}$$

$$\sum_{i=1}^{n} P_i^2 = \frac{1}{32}\left[9^{n+1} + 8n + 12(-3)^n - 12\right] \quad (\lambda = 3) \tag{17s}$$

$$\sum_{i=1}^{n} 2^{2i} L_i^2 = 2\left[2^{2n+1} F_{2n-1} - \frac{(-4)^{n+1} + 14}{5}\right] \tag{17'}$$

(18) $$\sum_{i=1}^{n}\left[(1+\lambda)O_i^2 + P_i^2\right] = \frac{(\lambda - 4)O_{2n} + \lambda(\lambda - 2)O_{2n-1} + 2 - \lambda}{\lambda - 3} \qquad (\lambda \neq 3)$$

(18s) $$\sum_{i=1}^{n}(4O_i^2 + P_i^2) = \frac{1}{16}\left[9^{n+1} + 8n - 9\right] \qquad \lambda = 3$$

(18') $$\sum_{i=1}^{n} 2^{2i}(5F_i^2 + L_i^2) = 8\left[2^{2n}F_{2n-1} - 1\right]$$

The proof of (16) follows:

$$\sum_{i=1}^{n} O_i^2 = \sum_{i=1}^{n}\left[\frac{A^i - B^i}{A - B}\right]^2$$

$$= \frac{1}{4(1+\lambda)}\left[\sum_{i=1}^{n} A^{2i} + \sum_{i=1}^{n} B^{2i} - 2\sum_{i=1}^{n}(AB)^i\right]$$

$$= \frac{1}{4(1+\lambda)}\left[\frac{A^2(A^{2n} - 1)}{A^2 - 1} + \frac{B^2(B^{2n} - 1)}{B^2 - 1} - 2\,\frac{AB[(AB)^n - 1]}{AB - 1}\right]$$

$$= \frac{1}{4(1+\lambda)}\left[\frac{(B^2 - 1)A^{2n+2} - A^2B^2 + A^2 + (A^2 - 1)B^{2n+2} - A^2B^2 + B^2}{A^2B^2 - A^2 - B^2 + 1} + 2R\right]$$

Since

$$A^2 - 1 = 1 + \lambda + 2\sqrt{1+\lambda}$$

and

$$B^2 - 1 = 1 + \lambda - 2\sqrt{1+\lambda}$$

we have

$$\sum_{i=1}^{n} O_i^2 = \frac{1}{4(1+\lambda)}\left[\frac{(1+\lambda)(A^{2n+2}+B^{2n+2}) - 2\sqrt{1+\lambda}(A^{2n+2}-B^{2n+2}) + A^2+B^2-2A^2B^2}{A^2B^2-A^2-B^2+1} + 2R\right]$$

$$= \frac{1}{2(1+\lambda)}\left[\frac{2(1+\lambda)P_{2n+2} - 4(1+\lambda)O_{2n+2} + 2P_2 - 2\lambda^2}{(\lambda+1)(\lambda-3)} + 2R\right] \qquad (\lambda \neq 3)$$

$$= \frac{1}{2(1+\lambda)}\left[\frac{P_{2n+2} - 2O_{2n+2} + 2 - \lambda}{\lambda - 3} + R\right] \qquad (\lambda \neq 3)$$

From (1) and (3) we get

$$P_{2n+2} = 2O_{2n+2} = (\lambda - 4)O_{2n} + \lambda(\lambda - 2)\ O_{2n-1}$$

Hence

$$\sum_{i=1}^{n} O_i^2 = \frac{1}{2(1+\lambda)}\left[\frac{(\lambda - 4)O_{2n} + \lambda(\lambda - 2)O_{2n-1} + 2 - \lambda}{\lambda - 3} + R\right] \ (\lambda \neq 3)$$

This completes the proof of (16).

We consider next identities involving the sums of products. The proof of the identity

$$(19) \qquad 2(1+\lambda)O_nO_m = P_{n+m} - (-\lambda)^m P_{n-m}$$

follows:

$$2(1+\lambda)O_nO_m = 2(1+\lambda)\left[\frac{A^n - B^n}{2\sqrt{1+\lambda}}\right]\cdot\left[\frac{A^m - B^m}{2\sqrt{1+\lambda}}\right]$$

$$= \frac{A^{n+m} + B^{n+m} - A^nB^m - A^mB^n}{2}$$

$$= \frac{A^{n+m} + B^{n+m}}{2} - \frac{A^mB^m(A^{n-m} - B^{n-m}}{2}$$

$$= P_{n+m} - (-\lambda)^m P_{n-m} .$$

From (19) we may write

$$2(1 + \lambda)O_2O_1 = P_3 - (-\lambda)P_1$$

$$2(1 + \lambda)O_3O_2 = P_5 - (-\lambda)^2P_1$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$2(1 + \lambda)O_{2n+1}O_n = P_{2n+1} - (-\lambda)^nP_1 \text{ .}$$

Adding these n equations gives

$$2(1 + \lambda)\sum_{i=1}^{n} O_iO_{i+1} = \sum_{i=1}^{n} P_{2i+1} - \sum_{i=1}^{n} (-\lambda)^iP_1$$

Using (13a) and the fact that $P_1 = 1$ we have

$$(20) \qquad 2(1 + \lambda)\sum_{i=1}^{n} O_iO_{i+1} = \frac{(\lambda - 4)O_{2n+1} + (\lambda - 2)O_{2n} + 4 - \lambda}{\lambda - 3} + R \qquad (\lambda \neq 3)$$

For the case $\lambda = 3$ we get

$$(20s) \qquad \sum_{i=1}^{n} O_iO_{i+1} = \frac{1}{128}\,[3^{2n+3} + 4(-3)^{n+1} - 8n - 15] \qquad (\lambda = 3)$$

For $\lambda = 4$ we have

$$(20') \qquad \sum_{i=1}^{n} 2^{2i}F_iF_{i+1} = \frac{4}{5}\left[2^{2n}F_{2n} + \frac{1}{5}\,[1 - (-4)^n]\right]$$

The proofs of the three following identities are left to the reader.

(21) $$2P_nP_m = P_{n+m} + (-\lambda)^m P_{n-m}$$

(22) $$2O_nP_m = O_{n+m} + (-\lambda)^m O_{n-m}$$

(23) $$2P_nO_m = O_{n+m} - (-\lambda)^m O_{n-m} .$$

Following the same procedure that was used above we arrive at the following identities:

(24) $$2\sum_{i=1}^{n} P_iP_{i+1} = \frac{(\lambda - 4)O_{2n+1} + \lambda(\lambda - 2)O_{2n} + 4 - \lambda}{\lambda - 3} - R \quad (\lambda \neq 3)$$

(24s) $$\sum_{i=1}^{n} P_iP_{i+1} = \frac{1}{32}\left[3^{2n+3} - 4(-3)^{n+1} - 8n - 39\right] \quad (\lambda = 3)$$

(24') $$\sum_{i=1}^{n} 2^{2i}L_iL_{i+1} = 4\left[2^{2n}F_{2n} - \frac{1}{5}\left[1 - (-4)^n\right]\right]$$

(25) $$2\sum_{i=1}^{n} P_{i+1}O_i = \frac{(\lambda - 4)P_{2n+1} + \lambda(\lambda - 2)P_{2n} - \lambda^2 + \lambda + 4}{(\lambda + 1)(\lambda - 3)} + R \quad (\lambda \neq 3)$$

(25s) $$\sum_{i=1}^{n} P_{i+1}O_i = \frac{1}{64}\left[3^{2n+3} + 8n - 24(-3)^n - 3\right] \quad (\lambda = 3)$$

(25') $$\sum_{i=1}^{n} 2^{2i}L_{i+1}F_i = \frac{4}{5}\left[2^{2n}L_{2n} - (-4)^n - 1\right]$$

(26) $$2\sum_{i=1}^{n} O_{i+1}L_i = \frac{(\lambda - 4)P_{2n+1} + \lambda(\lambda - 2)P_{2n} - \lambda^2 + \lambda + 4}{(\lambda + 1)(\lambda - 3)} - R \quad (\lambda \neq 3)$$

(26s) $$\sum_{i=1}^{n} O_{i+1}L_i = \frac{1}{64}\left[3^{2n+3} + 8n + 24(-3)^n - 51\right] \qquad (\lambda = 3)$$

(26') $$\sum_{i=1}^{n} 2^{2i}F_{i+1}L_i = \frac{4}{5}\left[2^{2n}L_{2n} - 3 + (-4)^n\right]$$

REFERENCE

1. cf. N. N. Vorob'ev, Fibonacci Numbers, pp. 15-20.

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(Continued from p. 369.)

Let the function h be defined by $h(s, t) = (3s + 4t, \ 2s + 3t)$. Using the method employed above, prove that all solutions in positive integers of Eq. (3) are given by

(17) $$(s_n, t_n) = h^n(1, 0), \quad n = 1, 2, 3, \cdots .$$

To be continued in the February issue of this Quarterly.

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[Continued from p. 384.]
according to the principles of a highly sophisticated harmonic system based on the canon of proportion of the Fibonacci Series: the system may yet prove to underlie other disparate aspects of Minoan design.[1]

[1]As it does design of structures elsewhere in the Aegean contemporary or later than Minoan palatial construction. There is evidence that the 1:1.6 ratio was employed in design previously in the Early Bronze Age in Greece and western Anatolia (disseration, loc. cit.).

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