NOTE ON A PAPER OF PAUL F. BYRD, AND A SOLUTION OF PROBLEM P-3

H. W. GOULD*
West Virginia University, Morgantown, W. Va.

Paul F. Byrd [1] has shown how to determine the coefficients c_k in the expansion

$$f(x) = \sum_{i=0}^{\infty} c_k \phi_{k+1}(x) \quad , \tag{1}$$

where f is an arbitrary power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad ,$$

and the polynomials $\phi_n(x)$ are defined by the recurrence

$$\phi_{n+2}(x) - 2x\phi_{n+1}(x) - \phi_n(x) = 0, \qquad \phi_0(x) = 0, \quad \phi_1(x) = 1, \tag{2}$$

or, equivalently, by the generating function

$$(1 - 2xt - t^2)^{-1} = \sum_{n=0}^{\infty} \phi_{n+1}(x)\, t^n \quad . \tag{3}$$

It is our object to point out that the expansion theory involved is a special case of a general treatment given by the author in [3]. In that paper the writer has studied generalized Humbert polynomials defined by the generating function

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, C) \, . \tag{4}$$

*Supported by National Sciences Foundation Research Grant GP-482.

(Received Nov. 1965)

The polynomials P_n include the polynomial systems of Louville, Legendre, Tchebycheff, Gegenbauer, Pincherle, Humbert, Kinney, Byrd, and several others. In particular, it is clear that

$$\phi_{n+1}(x) = P_n(2, x, -1, -1, 1) \quad . \tag{5}$$

It is shown in [3] that $P_n(m, x, y, p, C)$ satisfies

$$CnP_n - m(n - 1 - p)xP_{n-1} + (n - m - mp)yP_{n-m} = 0, \quad n \geq m \geq 1 \tag{6}$$

of which (2) is a corresponding special case.

It is also shown that

$$P_n(m, x, y, p, C) = \sum_{k=0}^{[n/m]} \binom{p}{k}\binom{p - k}{n - mk} C^{p-n+(m-1)k} y^k (-mx)^{n-mk} \tag{7}$$

with the corresponding inversion

$$\binom{p}{n}(-mx)^n = \sum_{k=0}^{[n/m]} (-1)^k \binom{p - n + k}{k} \frac{p + mk - n}{p + k - n} C^{n-k-p} y^k P_{n-mk}(m,x,y,p,C). \tag{8}$$

For Byrd's special case (5) these reduce to his relations

$$\phi_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n - k}{k} (2x)^{n-2k} \tag{9}$$

and

$$(2x)^n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \frac{n - 2k + 1}{n - k + 1} \phi_{n+1-2k}(x) \quad . \tag{10}$$

This, incidentally, solves his problem P-3 [1, page 29] and in a simpler manner than the complicated induction solution given in [4]. Actually, relations (7) and (8), and hence also (9) and (10), are special cases of the general inversion relations [2, (6.3), (6.4)] found by the writer:

$$F(n) = \sum_{k=0}^{[n/m]} \binom{p - n + mk}{k} f(n - mk) \tag{11}$$

<u>if and only if</u>

$$f(n) = \sum_{k=0}^{[n/m]} (-1)^k \binom{p - n + k}{k} \frac{p + mk - n}{p + k - n} F(n - mk) . \tag{12}$$

Proof of the reciprocal nature of (11) and (12) in turn depends upon general addition theorems for the binomial coefficients, typified by the relation

$$\sum_{k=0}^{n} (p + qk) A_k(a, b) A_{n-k}(c, b) = \frac{p(a + c) + qan}{a + c} A_n(a + c, b) , \tag{13}$$

where

$$A_k(a, b) = \frac{a}{a + bk} \binom{a + bk}{k} ,$$
$$a, b, c = \text{any real numbers.}$$

This relation actually was given in 1793 by Heinrich August Rothe in his Leipzig dissertation, and it is implied by relations in Lagrange's 1770 memoir on solution of equations. The reader may refer to a series of papers by the writer (since 1960) in the <u>Duke Mathematical Journal,</u> and to papers in the 1956 and 1957 volumes of the <u>American Mathematical Monthly</u>. Though not widely known, these general addition theorems enter into something on the order of several hundred papers in the literature. For example, a special case of (13) when $b = 4$ was used by Oakley and Wisner to enumerate classes of Flexagons.

We wish to note that relations (11) and (12) were used by the writer [2] to establish certain results about quasi-orthogonal number sets. The relations in [3] may be looked on as a generalization of the Fibonacci polynomials. Finally we note that Byrd's formula (4.4) for the coefficients c_k in (1) above are found in the limiting case from the corresponding expansion (6.9)-(6.10) found by the writer [3] for expressing an arbitrary polynomial as a linear combination of generalized Humbert polynomials. The formulas are too complicated to quote here.

REFERENCES

1. Paul F. Byrd, "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers," Fibonacci Quarterly, 1 (1963), No. 1, pp. 16-29. Note Problem P-3, p. 29.
2. H. W. Gould, "The Construction of Orthogonal and Quasi-Orthogonal Number Sets," Amer. Math. Monthly, 72 (1965), pp. 591-602.
3. H. W. Gould, "Inverse Series Relations and Other Expansions Involving Humbert Polynomials," Duke Math. J., Vol. 32, pp. 697-711, Dec. 1965.
4. Gary McDonald, Solution of Problem P-3, Fibonacci Quarterly, 3 (1965), pp. 46-48.
5. C. O. Oakley and R. J. Wisner, "Flexagons," Amer. Math. Monthly, 64 (1957), pp. 143-154.

* * * * *