CONVERGENCE OF THE COEFFICIENTS IN A RECURRING POWER SERIES

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1. INTRODUCTION

In this paper we use the following notation

$$\left(\sum_{w=0}^{\infty} c_w x^w\right)^k = \sum_{w=0}^{\infty} c_w^{(k)} x^w \quad ,$$

(For convenience, we shall write c_w instead of $c_w^{(1)}$.)

We define

$$\sum_{w=0}^{f} b_w x^w = F(x) \neq 0$$

for a finite f,

$$\sum_{w=0}^{t} a_w x^w = \prod_{w=1}^{m} (1 - r_w x)^{d_w} = Q(x)$$

for finite t and m, where the $d_w \neq 0$ and are positive integers. The $r_w \neq 0$ and are distinct and we say $|r_1|$ is the greatest $|r|$ in the $|r_w|$.

2. THEOREM 1

If

$$F(x)/Q(x) = \sum_{w=0}^{\infty} u_w x^w \quad ,$$

then

$$\lim_{n\to\infty} \left| u_n / u_{n-j} \right| \quad \text{(for a finite } j = 0, 1, 2, \cdots) \tag{2.1}$$

converges to $\left|r_1^j\right|$, where the $r_w \neq 0$ in $Q(x)$ are distinct with distinct moduli and $|r_1|$ is the greatest $|r|$ in the $|r_w|$.

Proof. It has been shown by Poincaré [1] that

$$\lim_{n\to\infty} u_n / u_{n-1} \tag{2.2}$$

converges to some root (r) in $Q(x)$. (We must then prove that this root (r) in $Q(x)$ is $|r_1|$.)

Let

$$M(x) = \prod_{w=1}^{m} (1 - r_w x)^{p_w} , \tag{2.3}$$

where the p_w are positive integers or $=0$ and

$$d_1 + p_1 = d_2 + p_2 = \cdots = p_w + d_w = k \qquad (k = 1, 2, 3, \cdots)$$

for a finite $w = 1, 2, 3, \cdots, m$.

Then,

$$M(x)Q(x) = \prod_{w=1}^{m} (1 - r_w x)^k = \phi_k(x) ,$$

so that

$$F(x)M(x)/Q(x)M(x) = F(x)M(x)/\phi_k(x) \tag{2.4}$$

$$= \sum_{w=0}^{\infty} u_w x^w = \sum_{w=0}^{\infty} c(k, w) x^w ,$$

where it is evident

$$u_n = c(k, n) \quad .$$

Now let

$$\phi_k(x) = \sum_{w=0}^{v} c_w^{(k)} x^w \qquad \text{(where } v \text{ is finite)} \quad ,$$

where combining this with (2.4), we write

$$\text{(2.5)} \qquad F(x)M(x)/\phi_{k-1}(x) = \sum_{w=0}^{\infty} c(k - 1, w)x^w$$

$$= \left(\sum_{w=0}^{v} c_w x^w\right)\left(\sum_{w=0}^{\infty} c(k, w)x^w\right) \quad ,$$

and combining coefficients leads to

$$\text{(2.5.1)} \qquad c(k - 1, n) = \sum_{w=0}^{v} c(k, n - w)c_w = \sum_{w=0}^{v} u_{n-w}\, c_w \quad ,$$

$$k = 2, 3, 4, \cdots \quad .$$

In (2.5.1), we replace k with $k + 1$ (where $k = 1, 2, 3, \cdots$) where combining this result with (2.2) leads to

$\lim_{n\to\infty} |c(k + 1, n)/c(k + 1, n - 1)|$ converges to some root (r) in $Q(x)$.

For convenience, we write the convergence as

$$\text{(2.5.2)} \qquad c(k + 1, n) = g_{k+1}\, c(k+1, n-1) \quad .$$

Combining (2.5.1) with k replaced by $k + 1$ with (2.5.2), it is easily shown, that for a finite v, we have

$$(2.5.3) \qquad c(k,n)/c(k,n-1) = g_k = \sum_{w=0}^{v} c(k+1, n-w)c_w \Big/ \sum_{w=0}^{v} c(k+1, n-w-1)c_w = g_{k+1} \;,$$

so that

$$(2.5.4) \qquad g_{k+1} = g_k = \cdots = g_1 \;.$$

Thus to complete the proof of Theorem 1, it remains to show that

$$|g_1| = |r_1| \;.$$

Then we consider the following (we refer to (2.3))

$$(2.6) \qquad (\phi(x))^{-1} = \prod_{w=1}^{m} (1 - r_w x)^{-1} = \sum_{w=0}^{\infty} e(m,w)x^w \quad \text{(for a finite } m)$$

for the convergence properties of $e(m,n)/e(m,n-1)$, where the $|r_w|$ are distinct and $|r_1|$ is the greatest root.

NOTE. For convenience, we write

$$e(m,n)/e(m,n-j) = r_1^j \qquad \text{(for a finite } j = 0, 1, 2, \cdots) \;,$$

in place of

$$\lim_{n \to \infty} |e(m,n)/e(m,n-j)| \text{ converges to } |r_1^j| \;.$$

For $m = 1$, we have

$$(2.7) \qquad (1 - r_1 x)^{-1} = \sum_{w=0}^{\infty} e(1, w)\, x^w ,$$

where

$$e(1, n) = r_1^n ,$$

so that

$$e(1, n)/e(1, n - j) = r_1^j .$$

For $m = 2$, we have

$$(2.8) \qquad [(1 - r_1 x)(1 - r_2 x)]^{-1} = \sum_{w=0}^{\infty} e(2, w)\, x^w .$$

where

$$e(2, n) = (r_1^{n+1} - r_2^{n+1})/(r_1 - r_2) ,$$

so that

$$e(2, n)/e(2, n - j) = r_1^j .$$

It now remains to consider for finite $m = 3, 4, 5, \cdots$, let

$$(2.9) \qquad \left(1 - \sum_{s=0}^{t-1} a_s x^{t-s}\right)^{-1} = \prod_{s=1}^{t} (1 - r_s x)^{-1} = 1 + \sum_{s=1}^{\infty} U_s x^s ,$$

for a finite $t = 3, 4, 5, \cdots$, where $U_0 = 1$.

Equating the coefficients in this leads to

$$(2.10) \qquad U_n = \sum_{s=1}^{t} a_{t-s} U_{n-s} \qquad (U_0 = 1) ,$$

and

$$U_1 = U_0 a_{t-1} , \quad U_2 = U_1 a_{t-1} + U_0 a_{t-2}, \cdots, U_t = \sum_{s=0}^{t-1} U_s a_s .$$

Also, since in (2.9), we have

$$\prod_{s=1}^{t} (1 - r_s x) = 1 - \sum_{s=0}^{t-1} a_s x^{t-s} ,$$

we may write

$$(2.11) \qquad \prod_{s=1}^{t} (x - r_s) = x^t - \sum_{s=0}^{t-1} a_s x^s = 0 .$$

We now combine (2.10) with (2.11) and write

$$(2.12) \qquad x^t = U_1 x^{t-1} + \sum_{s=2}^{t} \left(U_s - \sum_{r=1}^{s-1} U_r a_{t+r-s} \right) x^{t-s} .$$

Multiplying (2.12) by x and combining the result with

$$U_1 x^t = U_1 \sum_{s=0}^{t-1} a_s x^s$$

in (2.11) leads to

$$(2.13) \qquad x^{t+1} = U_2x^{t-1} + \sum_{r=0}^{t-3}\left(U_{r+3} - \sum_{s=0}^{r} U_{r+2-s}a_{t-s-1}\right)x^{t-r-2} + U_1a_0 .$$

Now, multiplying (2.13) by x and combining the result with

$$U_2x^t = U_2\left(\sum_{s=0}^{t-1} a_sx^s\right)$$

in (2.11), we then have

$$(2.14) \qquad x^{t+2} = U_3x^{t-1} + \sum_{r=0}^{t-3}\left[\left(U_{r+4} - \sum_{s=0}^{r} U_{r+3-s}a_{t-s-1}\right)x^{t-r-2}\right] + a_0U_2 .$$

We continue in the exact way we found (2.13) and (2.14) for $n - 1$ steps to get

$$(2.15) \qquad x^{t+n-1} = U_nx^{t-1} + \sum_{r=0}^{t-3}\left[\left(U_{n+r+1} - \sum_{s=0}^{r} U_{n+r-s}a_{t-s-1}\right)x^{t-r-2}\right] + U_{n-1}a_0 = U_nx^{t-1} + R(x) + U_{n-1}a_0 .$$

We now continue (2.15) with (2.11) to get the following t equations

$$(2.16) \qquad \begin{aligned} r_1^{t+n-1} &= U_nr_1^{t-1} + R(r_1) + U_{n-1}a_0 , \\ &\ldots\ldots\ldots\ldots , \\ r_t^{t+n-1} &= U_nr_t^{t-1} + R(r_t) + U_{n-1}a_0 . \end{aligned}$$

Next, we consider the t equations obtained from (2.16). These t equations in the t unknown can be solved by Cramer's rule to obtain

$$U_n D_2 = D_1(n) \, , \tag{2.17}$$

where $D_1(n)$ and D_2 are the determinants given below:

$$D_1(n) = \begin{vmatrix} r_1^{t+n-1} & r_1^{t-2} & \cdots & r_1 & 1 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ r_t^{t+n-1} & r_t^{t-2} & \cdots & r_t & 1 \end{vmatrix} \tag{2.18}$$

$$D_2 = \begin{vmatrix} r_1^{t-1} & r_1^{t-2} & \cdots & r_1 & 1 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ r_t^{t-1} & r_t^{t-2} & \cdots & r_t & 1 \end{vmatrix} \tag{2.19}$$

We now replace n with n - 1 in (2.17) to get

$$U_{n-1} D_2 = D_1(n-1) \, , \tag{2.20}$$

and dividing (2.17) by (2.20), we get

$$U_n / U_{n-1} = D_1(n) / D_1(n-1) \, . \tag{2.21}$$

Since the $r_t \neq 0$ and are distinct, then one root (say $|r_1|$ is greater than the other roots, and we write

$$U_n / U_{n-1} = (D_1(n) / r_1^{t+n-2}) / (D_1(n-1) / r_1^{t+n-2}) \tag{2.22}$$

Now in (2.22) we let r_1^{t+n-2} (in the numerator) divide every term of the first column in (2.18) and r_1^{t+n-2} (in the denominator) divide every term in the first column of (2.18) (with n replaced by n - 1). Then if we let $n \to \infty$ it is evident that

(2.23) $$\lim_{n \to \infty} \left| U_n / U_{n-1} \right| = \left| r_1 \right| .$$

Now for a finite t we write

$$\lim_{n \to \infty} \left| U_{n-j} / U_{n-j-1} \right| = \left| r_1 \right| \quad (j = 0, 1, 2, \cdots, t - 1) ,$$

so that

(2.24) $$\lim_{n \to \infty} \left| U_n / U_{n-t} \right| = \left| r_1^t \right| .$$

Multiplying the F(x) in (1) with

$$\sum_{s=0}^{\infty} U_s x^s$$

in (2.9), we write

(2.25) $$\left(\sum_{w=0}^{f} b_w x^w \right) \left(\sum_{s=0}^{\infty} U_s x^s \right) = \sum_{s=0}^{\infty} C_s x^s ,$$

where comparing the coefficients we have

(2.26) $$C_n = \sum_{s=0}^{f} U_{n-s} b_s .$$

Now, since f is finite, and by the results in (2.23), we write

$$C_n = r_1 \sum_{s=0}^{f} U_{n-s-1} b_s = r_1 C_{n-1} ,$$

where combining this with the $r_t \neq 0$ and are distinct (so that we may add that the r_t have distinct moduli), leads to the completion of the proof for Theorem 1.

From (2.7), (2.8), and (2.17), the following corollary is immediate:

Corollary. If

$$\prod_{s=1}^{t} (1 - r_s x)^{-1} = \sum_{s=0}^{\infty} U_s x^s \quad (U_0 = 1),$$

where the $r_s \neq 0$ and are distinct, then

(2.27) It is always possible to solve for the U_n $(n = 0, 1, 2, \cdots)$ as a function of the r_s.

SECTION 3

Let

$$\left(1 - \sum_{w=1}^{t} a_w x^w\right)^{-k} = \prod_{w=1}^{t} (1 - r_w x)^{-k} = \sum_{w=0}^{\infty} c_w^{(k)} x^w$$

$(c_0^{(k)} = 1$ and $k = 1, 2, 3, \cdots)$ for a finite $t = 2, 3, 4, \cdots$ and the given roots $r_w \neq 0$ and are distinct. We also define

$$S(x) = \sum_{w=1}^{t} \sum_{r=w}^{t} a_r c_{n+w-r} x^{w-1} = 0$$

and

$$b = \sum_{w=2}^{t} a_w x_1^{w-2}$$

where ${}^*x_1 \neq 0$ and is a root in $S(x) = S(x_1) = 0$.

We then have the following:

Theorem 2. If

$$c_0 = 1, \quad c_1 = a_1c_0, \quad c_2 = a_1c_1 + a_2c_0, \cdots$$

$$\cdots, c_t = \sum_{w=0}^{t-1} a_{w+1}\, c_{t-w-1}$$

and

$$p_j = a_1(k + n - j) \qquad (j = 1, 2, 3, \cdots, n) \quad ,$$

$$q_{m+1} = b(n - m)(2k + n - m - 1)$$
$$(m = 1, 2, 3, \cdots, n-1)$$

then

(3.1) $$nc_n^{(k)} / c_{n-1}^{(k)} = E_n / G_n \qquad (k, n = 1, 2, 3, \cdots) \quad ,$$

where E_n and G_n are the determinants given below.

(3.1.1) $$E_n = \begin{vmatrix} p_1 & q_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & p_2 & q_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & p_3 & q_4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & p_4 & q_5 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & p_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & p_n \end{vmatrix}$$

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*It should be noted that since the a's are constant for a fixed t, that the root x_1 will be determined as a variable, since it is a function of the c_n and will, of course, change values for different n.

$$(3.1.2)\qquad G_n = \begin{vmatrix} p_2 & q_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & p_3 & q_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & p_4 & q_5 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & p_5 & q_6 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & p_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & p_n \end{vmatrix}$$

Proof. Let

$$(3.2)\qquad 1 = \left(1 - \sum_{w=1}^{t} a_w x^w\right)\left(\sum_{w=0}^{n} c_w x^w\right) \quad \text{(for a finite } n) ,$$

where the a_w and the c_w are identical to those in (3). Then multiplying and combining the terms in (3.2) leads to $S(x_1) = S(x) = 0$ in (3).

Now, taking each side of (3.2) to the k^{th} power, we write

$$(3.3)\qquad 1^k = \left(1 - \sum_{w=1}^{t} a_w x^w\right)^k \left(\sum_{w=0}^{n} c_w^{(k)} x^w + J(x)\right) \qquad (k = 2, 3, \cdots),$$

(where, of course, x_1 is a root in (3.3)).

Using the corresponding values in (3), we write (3.3) as

$$(3.3.1)\qquad 1 = (1 - a_1 x - bx^2)^k \left(\sum_{w=0}^{n} c_w^{(k)} x^w + J(x)\right)$$

Differentiation of (3.3.1) leads to

$$k(a_1 x + 2bx^2)\left(\sum_{w=0}^{n} c_n^{(k)} + J(x)\right) = (1 - a_1 x - bx^2)\left(\sum_{w=1}^{n} n c_n^{(k)} x^n + W(x)\right)$$

and by comparing coefficients, we conclude that

$$(3.4)\qquad nc_n^{(k)} = a_1(k + n - 1)c_{n-1}^{(k)} + b(2k + n - 2)c_{n-2}^{(k)}$$

for

$$k = 2, 3, \cdots, \quad n = 2, 3, \cdots, \quad c_0^{(k)} = 1 \text{ and } c_1^{(k)} = a_1 k .$$

When we divide (3.4) by $c_{n-1}^{(k)}$, we get

$$\frac{nc_n^{(k)}}{c_{n-1}^{(k)}} = a_1(k + n - 1) + \frac{b(2k + n - 2)(n - 1)}{\dfrac{(n - 1)c_{n-1}^{(k)}}{c_{n-2}^{(k)}}} \qquad (n, k = 2, 3, \cdots) ,$$

which in turn, along with $c_0^{(k)} = 1$ and $c_1^{(k)} = a_1 k$, implies (along with the values of p and q in (3)),

$$(3.5)\qquad \frac{nc_n^{(k)}}{c_{n-1}^{(k)}} = p_1 + \frac{q_2}{p_2} + \frac{q_3}{p_3} + \cdots + \frac{q_{n-1}}{p_{n-1}} + \frac{q_n}{p_n} = K(n) .$$

We complete the proof of Theorem 2 with Euler's statement [2]

$$K(n) = E_n / G_n ;$$

and we resolve for the case when $k = 1$ with (2.27).

Corollary. In

$$\prod_{w=1}^{t} (1 - r_w x)^{-k} = \left(1 - \sum_{w=1}^{t} a_w x^w\right)^{-k} = 1 + \sum_{w=1}^{\infty} c_w^{(k)} x^w ,$$

it is always possible to solve for

$$nc_n^{(k)} / c_{n-1}^{(k)} = K(n) = E_n / G_n \quad (k \text{ and } n = 2, 3, \cdots) \tag{3.6}$$

when $t = 2, 3, 4,$ or $5,$ if the $r_w \neq 0$ and are distinct.

Proof. In (2.27), it is seen that the c_n maybe determined. Now, since $t - 1 = 1, 2, 3,$ or $4,$ then the roots (each root is a function of the c_n) in $S(x)$ (in 3) may always be found, so that we will obtain values for the p and q. We then complete the proof of the corollary by observing that E_n and G_n are both functions of the p and q.

In conclusion: We solve when $t = 1$ and we write

$$(1 - r_x)^{-k} = \sum_{w=0}^{\infty} d_w^{(k)} x^w \qquad (d_0^{(k)} = 1, \quad r \neq 0) .$$

Now, differentiating, we have

$$xkr\left(\sum_{w=0}^{\infty} d_w^{(k+1)} x^w\right) = \sum_{w=1}^{\infty} wd_w^{(k)} x^w$$

and comparing the coefficients leads to

$$nd_n^{(k)} = d_{n-1}^{(k+1)} r^k$$

so that

$$\prod_{w=1}^{n} wd_w^{(k+n-w)} = r^n \prod_{w=0}^{n-1} (k + n - w - 1) d_w^{(k+n-w)}$$

and we then have

$$d_n^{(k)} = r^n (k + n - 1)! / n! \, (k - 1)! \quad .$$

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[Continued from p. 40.]

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