THE BRACKET FUNCTION AND FONTENÉ-WARD GENERALIZED BINOMIAL COEFFICIENTS WITH APPLICATION TO FIBONOMIAL COEFFICIENTS

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INTRODUCTION

In 1915 Georges Fontené (1848-1928) published a one-page note [4] suggesting a generalization of binomial coefficients, replacing the natural numbers by an arbitrary sequence A_n of real or complex numbers. He gave the fundamental recurrence relation for these generalized coefficients and noted that for $A_n = n$ we recover the ordinary binomial coefficients, while for $A_n = q^n - 1$ we obtain the q-binomial coefficients studied by Gauss (as well as Euler, Cauchy, F. H. Jackson, and many others later).

These generalized coefficients of Fontené were later rediscovered by the late Morgan Ward (1901-1963) in a short but remarkable paper [16] in 1936 which developed a symbolic calculus of sequences. He does not mention Fontené. Failing to find other pioneers we shall call the generalized coefficients Fontené-Ward generalized binomial coefficients. We avoid the symbolic method of Ward in our work.

Since 1964, there has been an accelerated interest in Fibonomial coefficients. These correspond to the choice $A_n = F_n$, where F_n is the Fibonacci number defined by

$$F_{n+1} = F_n + F_{n-1}$$

with

$$F_0 = 0, \quad F_1 = 1 .$$

This idea seems to have originated with Dov Jarden [11] in 1949. He actually states the more general definition but only considers the Fibonomial case. Fibonomial coefficients have been quite a popular subject in this Quarterly since 1964 as references [1], [9], [10], [13], and [15] will tell. See also [17].

Because of the restricted nature of the three special cases of Fontené-Ward coefficients cited above, and because so many properties may be obtained in the most general case, we shall develop below a number of very striking general theorems which include a host of special cases among the references at the end of this paper. Despite an intensive study of all available books and journals for twenty years, it is possible that some of our results have been anticipated or extended. Indeed certain notions below are familiar in variant form and we claim only a novel presentation of what seems obvious. However a large body of the results below extend apparently new results of the author [7], [8] and we obtain the following elegant general results: Representation of Fontené-Ward coefficients as a linear combination of greatest integer (bracket function) terms; Representation of the bracket function as a linear combination of Fontené-Ward coefficients; A Lambert series expansion of a new number-theoretic function; A powerful inversion theorem for series of Fontené-Ward coefficients; and some miscellaneous identities including a brief way to study Fontené-Ward multinomial coefficients by avoiding a tedius argument of Kohlbecker [13].

The present paper originated out of discussions with my colleagues, Professors R. P. Agarwal and A. M. Chak, about the feasibility of extending Ward's ideas to broader areas of analysis and number theory. Chak [3] has developed and applied Ward's symbolic calculus of sequences to discuss numerous generalized special functions.

Every result below can be immediately applied to the Fibonacci triangle, or new variants thereof, and the inversion theorem given below is expected to be especially useful to Fibonacci enthusiasts. Such inversion theorems are valuable tools in analysis and have not been previously introduced or applied for Fibonomial coefficients. We may even take our sequence A_n to be the non-Fibonacci numbers and study a non-Fibonomial triangle.

FONTENÉ-WARD COEFFICIENTS: DEFINITION AND PROPERTIES

By the Fontené-Ward generalized binomial coefficient with respect to a sequence A_n we shall mean the following:

$$(1) \qquad \left\{ {n \atop k} \right\}_A = \left\{ {n \atop k} \right\} = \frac{A_n A_{n-1} \cdots A_{n-k+1}}{A_k A_{k-1} \cdots A_1}, \quad \text{with } \left\{ {n \atop 0} \right\} = \left\{ {n \atop n} \right\} = 1 ,$$

and we also require that

$$\left\{ {n \atop k} \right\} = 0 \qquad \text{whenever} \qquad k < 0 \text{ or } k > n . \tag{2}$$

The sequence A_n is essentially arbitrary but we do require that $A_0 = 0$ and $A_n \neq 0$ for $n \geq 1$. Ward [16] took $A_1 = 1$, and there is no loss of generality in doing that. However we cannot in general simplify very much and we shall retain A_1 as arbitrary. One has only to multiply Ward's sequence by A_1 to obtain our results. When no confusion can occur as to our choice of the basic sequence A_n we shall omit the subscript A in our notation (1). We use braces to set our coefficients apart from ordinary and q-binomial coefficients.

With this definition we can now exhibit the Fontené-Ward Triangle:

$$\begin{array}{c}
1 \\
1 \qquad 1 \\
1 \qquad \dfrac{A_2}{A_1} \qquad 1 \\
1 \qquad \dfrac{A_3}{A_1} \qquad \dfrac{A_3}{A_1} \qquad 1 \\
1 \qquad \dfrac{A_4}{A_1} \qquad \dfrac{A_4A_3}{A_1A_2} \qquad \dfrac{A_4}{A_1} \qquad 1 \\
1 \qquad \dfrac{A_5}{A_1} \qquad \dfrac{A_5A_4}{A_1A_2} \qquad \dfrac{A_5A_4}{A_1A_2} \qquad \dfrac{A_5}{A_1} \qquad 1 \\
1 \qquad \dfrac{A_6}{A_1} \qquad \dfrac{A_6A_5}{A_1A_2} \qquad \dfrac{A_6A_5A_4}{A_1A_2A_3} \qquad \dfrac{A_6A_5}{A_1A_2} \qquad \dfrac{A_6}{A_1} \qquad 1 \\
1 \qquad \dfrac{A_7}{A_1} \qquad \dfrac{A_7A_6}{A_1A_2} \qquad \dfrac{A_7A_6A_5}{A_1A_2A_3} \qquad \dfrac{A_7A_6A_5}{A_1A_2A_3} \qquad \dfrac{A_7A_6}{A_1A_2} \qquad \dfrac{A_7}{A_1} \qquad 1
\end{array}$$

It is evident that the triangle is symmetrical in the sense that

$$\left\{ {n \atop k} \right\} = \left\{ {n \atop n-k} \right\} \qquad 0 \leq k \leq n \tag{3}$$

We can make the definition (1) more symmetrical by introducing generalized factorials. We can define

(4) $$\left\{ {n \atop k} \right\} = \frac{[n]!}{[k]!\,[n-k]!} \quad ,$$

where

$$[n]! = A_n A_{n-1} \cdots A_2 A_1 \quad \text{with} \quad [0]! = 1 \, .$$

This is equivalent to the previous definition and allows us to adapt a number of familiar binomial coefficient identities to our study. For example, it is clear that we have

(5) $$\left\{ {n \atop k} \right\} \left\{ {k \atop j} \right\} = \left\{ {n \atop j} \right\} \left\{ {n-j \atop k-j} \right\} \quad ,$$

which we shall need later.

The basic recurrence relation for the Fontené-Ward coefficients was given by Fontené and is as follows:

(6) $$\left\{ {n \atop k} \right\} - \left\{ {n-1 \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} \frac{A_n - A_{n-k}}{A_k} \quad ,$$

In this, change k to $n - k$ and apply (3). We find that

(7) $$\left\{ {n \atop k} \right\} - \left\{ {n-1 \atop k-1} \right\} = \left\{ {n-1 \atop k} \right\} \frac{A_n - A_k}{A_{n-k}} \quad .$$

In general $A_n - A_k \neq A_{n-k}$. The fraction does equal 1 when we set $A_j = j$, and the fraction equals q^k when we set $A_j = (q^j - 1)/(q - 1)$. Fontené is correct that we get q-binomial coefficients with $A_j = q^j - 1$, but it is better to include the factor $q - 1$ in the denominator so that we can also assert that

$$\lim_{q \to 1} A_j = j$$

making the q-case then agree with ordinary natural numbers.

In the Fibonomial coefficient case, when $A_k = F_k$, write

(8)
$$f(n,k) = \frac{F_n - F_k}{F_{n-k}} .$$

It is easily verified that f satisfies the recurrence

(9)
$$f(n + 1,\ k + 1) = f(n,k) + f(n - 1, k - 1) .$$

By induction it then follows that

(10)
$$f(n + r,\ k + r) = F_{r+1}f(n,k) + F_r f(n-1,k-1) .$$

From this one may easily find

(11)
$$f(n,k) = F_k \frac{F_{n-k+1} - 1}{F_{n-k}} + F_{k-1}$$

which may also be derived directly from (8) and the relation

(12)
$$F_n = F_k F_{n-k+1} + F_{k-1}F_{n-k} .$$

There are then an abundance of ways to modify $f(n,k)$ using known Fibonacci relations, and the particular way we might interpret $f(n,k)$ determines the nature of the Fibonomial relations which will follow from our general theorems.

An important observation is this: $f(n,k)$ is independent of n in the case of ordinary and q-binomial coefficients, but not in the Fibonomial case. This makes the possibility of having certain expansions generalize depend on the way in which we can modify the recurrence.

We return to relation (6) and sum both sides with respect to the upper index. Clearly we obtain the relation

(13)
$$\sum_{j=k}^{n} \frac{A_j - A_{j-k}}{A_k} \left\{ \begin{matrix} j - 1 \\ k - 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} ,$$

which is the analogue of the familiar formula

$$\sum_{j=k}^{n} \binom{j-1}{k-1} = \binom{n}{k} .$$

Relation (13) will be very important to us in what follows.

We next define Fontené-Ward multinomial coefficients in the obvious way:

(14) $$\left\{ {n \atop k_1, k_2, \cdots, k_r} \right\} = \frac{\prod_{i=1}^{n} A_i}{\prod_{i=1}^{k_1} A_i \cdot \prod_{i=1}^{k_2} A_i \cdots \prod_{i=1}^{k_r} A_i} ,$$

subject to $n = k_1 + k_2 + \cdots + k_r$. For $A_i = i$ these pass over to the ordinary multinomial coefficients. What is more, (14) satisfies the following special relation: Set $r = 2$ and write $k_1 = a$, $k_2 = b$ with $a + b = n$. Then

(15) $$\left\{ {n \atop a,b} \right\} = \left\{ {n \atop a} \right\}$$

in terms of our original definition (1). Moreover, trinomial and higher order coefficients are products of ordinary Fontené-Ward generalized binomial coefficients:

(16) $$\left\{ {n \atop a,b,c} \right\} = \left\{ {n \atop a} \right\} \left\{ {n-a \atop b} \right\} , \quad a + b + c = n$$

(17) $$\left\{ {n \atop a,b,c,d} \right\} = \left\{ {n \atop a} \right\} \left\{ {n-a \atop b} \right\} \left\{ {n-a-b \atop c} \right\} , \quad a+b+c+d = n ,$$

(18) $$\left\{ {n \atop a,b,c,d,e} \right\} = \left\{ {n \atop a} \right\} \left\{ {n-a \atop b} \right\} \left\{ {n-a-b \atop c} \right\} \left\{ {n-a-b-c \atop d} \right\} , a+b+c+d+e = n,$$

and the general result follows at once by induction. This is a well-known device for ordinary multinomial coefficients and the application here is that once one proves that the Fontené-Ward binomial coefficient is an integer for some sequence A_n, then the Fontené-Ward multinomial coefficients, by the above relations, are integers, being just products of integers. This circumvents the tedius argument of Kohlbecker [13] for multinomial Fibonomial coefficients, for example.

Making use of the ideas developed so far and paralleling the steps in a previous paper [8], we are now in a position to state and prove our first major result. We have

Theorem 1. The Fontené-Ward generalized binomial coefficient may be expressed as a linear combination of bracket functions by the formula

$$\left\{ {n \atop k} \right\} = \left[\frac{n}{k}\right] + \sum_{j=k+1}^{n} \left[\frac{n}{j}\right] R_k(j, A) = \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j, A) , \tag{19}$$

where the number-theoretic function R is defined by

$$R_k(j, A) = \sum_{d|j} \frac{A_d - A_{d-k}}{A_k} \left\{ {d - 1 \atop k - 1} \right\} \mu(j/d) , \tag{20}$$

with $\mu(n)$ being the ordinary Moebius function in number theory.

Proof. Again we use the formula of Meissel

$$\sum_{m \le x} \left[\frac{x}{m}\right] \mu(m) = 1 , \quad x \ge 1 ,$$

and apply this to formula (13) precisely as was done in [8]. The result follows at once. It is easily seen that $R_k(k, A) = 1$. There will be no confusion of $R_k(j, A)$ with $R_k(j, q)$ in the former paper if we merely make a convention that whenever we have a sequence we denote it by a capital letter and then (20) is meant. Thus $R_k(j, F)$ would mean the Fibonomial case. Thus our first theorem expands the Fibonomial coefficient as a linear combination of bracket functions.

The expansion inverse to this requires a little more care. It was found in [8] by means of a certain inversion theorem for q-binomial coefficients. We must pause and establish the corresponding inversion principle for the most general case. Suppose we set

$$f(n) = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} g(k) \quad . \tag{21}$$

Then we find: $g(0) = f(0)$.

$$f(1) = -g(0) + g(1) \ ,$$

whence

$$g(1) = f(1) + f(0) \ .$$

Then

$$f(2) = g(0) - \frac{A_2}{A_1} g(1) + g(2) \ ,$$

from which we find

$$g(2) = f(2) + \frac{A_2}{A_1} f(1) + \left(\frac{A_2}{A_1} - 1 \right) f(0) \ .$$

Similarly it is easily found that

$$\begin{aligned} g(3) &= f(3) + \frac{A_3}{A_1} f(2) + \frac{A_3}{A_1}\left(\frac{A_2}{A_1} - 1\right) f(1) + \left(1 + \frac{A_3}{A_1}\left(\frac{A_2}{A_1} - 2\right)\right) f(0) \\ &= \left\{ {3 \atop 0} \right\} f(3) + \left\{ {3 \atop 1} \right\} f(2) + \left\{ {3 \atop 2} \right\} B_2 f(1) + \left\{ {3 \atop 3} \right\} B_3 f(0) \ , \end{aligned}$$

and it appears that the B_k are independent of n and any number may be found in succession. This is quite correct, for we may readily solve the system of equations necessary to determine such B_k coefficients as will invert (21).

The next step gives

$$B_4 = -1 + \frac{A_4}{A_1}\left(2 - \frac{A_3}{A_2}\left(\frac{A_2}{A_1} - 1\right) + \frac{A_3}{A_1}\left(\frac{A_2}{A_1} - 2\right)\right).$$

Put

$$(22) \qquad g(n) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} B_k f(n-k) \;, \quad \text{with } B_0 = B_1 = 1, \; B_2 = A_2/A_1 - 1, \text{ etc.}$$

It is easily seen by an inductive argument that B_k is independent of f and n. On the one hand, (22) would require us to have

$$(23) \qquad g(n+1) = \sum_{k=0}^{n+1} \left\{ {n+1 \atop k} \right\} B_k f(n+1-k) = \sum_{j=0}^{n+1} \left\{ {n+1 \atop j} \right\} B_{n+1-j} f(j) \;.$$

On the other hand we have from (21) that

$$f(n+1) = g(n+1) + \sum_{k=0}^{n} (-1)^{n+1-k} \left\{ {n+1 \atop k} \right\} g(k) \;,$$

whence

$$g(n+1) = f(n+1) - \sum_{k=0}^{n} (-1)^{n+1-k} \left\{ {n+1 \atop k} \right\} g(k) =$$

$$= f(n+1) + \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n+1 \atop k} \right\} \sum_{j=0}^{k} \left\{ {k \atop j} \right\} B_j f(k-j) \;,$$

by (22),

$$= f(n+1) + \sum_{j=0}^{n} f(j) \sum_{k=j}^{n} (-1)^{n-k} \left\{ {n+1 \atop k} \right\} \left\{ {k \atop j} \right\} B_{k-j} \;.$$

This expansion must agree with (23) if the induction is to proceed, so we equate coefficients of $f(j)$ to determine a recurrence relation for B_k. At the same time we apply the identity (5) and we have the result that

$$B_{n+1-j} = \sum_{k=j}^{n} (-1)^{n-k} \left\{ {n+1-j \atop k-j} \right\} B_{k-j}\,, \quad \text{for } 0 \le j \le n\,. \tag{24}$$

In particular set $j = 0$. We find the remarkably simple recurrence

$$B_{n+1} = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n+1 \atop k} \right\} B_k, \text{ valid for } n \ge 0\,. \tag{25}$$

From this it is easily seen that we can summarize our recurrence for B_n in the single formula

$$\sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} B_k = \begin{cases} 1, & \text{for } n = 0\,, \\ 0, & \text{for } n \ge 1\,. \end{cases} \tag{26}$$

This in turn can be given a handy symbolic expression

$$\{B - 1\}^n = \delta_1^n = \begin{cases} 1, & n = 0 \\ 0, & n \ge 1 \end{cases}\,, \tag{27}$$

if we just adopt an umbral binomial theorem that

$$\{x + y\}^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x_k y_{n-k}\,.$$

We shall next evaluate the B coefficients explicitly.

The sequence B_n is determined uniquely by the relation (26), and we can easily solve this by means of determinants. The result of this can be put in the form

$$
(28)\qquad B_n = (-1)^n \begin{vmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -\genfrac\{\}{0pt}{}{2}{1} & 1 & 0 & \cdots & 0 \\ -1 & \genfrac\{\}{0pt}{}{3}{1} & -\genfrac\{\}{0pt}{}{3}{2} & 1 & \cdots & 0 \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdots, & \genfrac\{\}{0pt}{}{n-1}{n-2} & 1 \\ (-1)^n, & -(-1)^n \genfrac\{\}{0pt}{}{n}{1}, & (-1)^n \genfrac\{\}{0pt}{}{n}{2}. & \cdots, & \genfrac\{\}{0pt}{}{n}{n-2}, & -\genfrac\{\}{0pt}{}{n}{n-1} \end{vmatrix}
$$

valid for $n \geq 1$.

The n-by-n determinant and the recurrences (25)-(26) allow us to compute as many B's as needed.

It was no accident that we write (26) as (27) and as a Kronecker delta, for not only does (26) allow us to invert (21) to obtain (22), but the converse is also true, (26) allows us to invert (22) back to (21). We have in fact

Theorem 2. For sequences f and g,

$$
(29)\qquad f(n) = \sum_{k=0}^{n} (-1)^{n-k} \genfrac\{\}{0pt}{}{n}{k} g(k)
$$

if and only if

$$
(30)\qquad g(n) = \sum_{k=0}^{n} \genfrac\{\}{0pt}{}{n}{k} B_{n-k} f(k) \quad ,
$$

where B_k satisfies recurrence (26), and is given explicitly by (28).

To illustrate the proof we will show that (30) implies (29), assuming (26). We have

$$
\sum_{k=0}^{n} (-1)^{n-k} \genfrac\{\}{0pt}{}{n}{k} g(k) = \sum_{k=0}^{n} (-1)^{n-k} \genfrac\{\}{0pt}{}{n}{k} \sum_{j=0}^{k} \genfrac\{\}{0pt}{}{k}{j} B_{k-j} f(j) \ ,
$$

$$
= \sum_{j=0}^{n} \genfrac\{\}{0pt}{}{n}{j} f(j) \sum_{k=0}^{n-j} (-1)^{n-j-k} \genfrac\{\}{0pt}{}{n-j}{k} B_k = f(n) \quad ,
$$

as required, for relation (26)-(27) is equivalent to the Kronecker delta

$$\sum_{k=0}^{n-j} (-1)^{n-j-k} \left\{ {n-j \atop k} \right\} B_k = \delta_j^n . \tag{31}$$

The reader should have no difficulty in showing that (29) implies (30), relation (31) again being what is needed to cancel out unwanted terms.

These relations are nothing more than extensions of the familiar inversions given in [6], [7], [8].

The application and use of Theorem 2 for Fibonomial expansions needs little elaboration. It allows often to solve for something given implicitly under the summation sign.

As was done in [6] and [8] we need some small variations of Theorem 2. It is easy to see that the theorem can be stated in the equivalent form

$$f(n) = \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\} g(k) \tag{32}$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\} B_{n-k} f(k) . \tag{33}$$

And we also have

$$f(n) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} g(k) \tag{34}$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} B_{n-k} f(k) \quad . \tag{35}$$

It is this last form of our theorem that will be used now to find an expansion inverse to Theorem 1. Our steps are the same as in [8].

Theorem 3. The bracket function may be expressed as a linear combination of Fontené-Ward generalized binomial coefficients by the formula

$$\left[\frac{n}{k} \right] = \left\{ {n \atop k} \right\} + \sum_{j=k+1}^{n} \left\{ {n \atop j} \right\} Q_k(j, A) = \sum_{j=k}^{n} \left\{ {n \atop j} \right\} Q_k(j, A) \quad , \tag{36}$$

where the coefficients $Q_k(j, A)$ are given by

$$Q_k(j, A) = \sum_{d=k}^{j} (-1)^{j-d} \left\{ {j \atop d} \right\} \left[\frac{d}{k} \right] B_{j-d} \quad , \tag{37}$$

and the B's are given by (26)-(28).

Proof. Assume expansion (36) for unknown Q's. Then by the inversion pair (34)-(35), with $f(n) = [n/k]$ and $g(n) = Q_k(n, A)$, and writing j for k in (34)-(35), the result is immediate.

Hence as a Fibonacci item, this theorem allows one to express the bracket function in terms of Fibonomial coefficients.

The next order of work in [8] was to see if the two expansions, bracket in terms of binomial and conversely, implied a more general inversion theorem; i.e., whether we can now show that our coefficients R and Q are orthogonal in general. Our success in doing this would depend on getting the Lambert series for R and an inverse series for Q. The binomial theorem was used to obtain the latter in [8] and this expansion, the binomial theorem, is more troublesome in our general situation. However we can obtain next the Lambert series for R.

Let us note a general series lemma: For a function $f = f(x,y)$,

$$\sum_{n=1}^{\infty} \sum_{d|n} f(d, n) = \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} f(d, md) \tag{38}$$

This is merely the limiting case of relation (20) in [7] for example.

Theorem 4. The Lambert series expansion for $R_k(j, A)$ is given by

$$\sum_{j=k}^{\infty} R_k(j, A) \frac{x^j}{1 - x^j} = \sum_{n=k}^{\infty} \frac{A_n - A_{n-k}}{A_k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} x^n . \tag{39}$$

Proof. First of all the ordinary Moebius inversion theorem applied to relation (20) inverts this to yield

$$\frac{A_n - A_{n-k}}{A_k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} = \sum_{d|n} R(d, A) , \tag{40}$$

which may itself be looked on as a valuable expansion of the Fontené-Ward generalized binomial coefficients in terms of the function $R_k(d, a)$. This is merely the generalization of the combinatorial formula

$$C_k(n) = \binom{n-1}{k-1} = \sum_{d|n} R_k(d) ,$$

found in [7].

Multiply (40) through by x^n and sum both sides on n. We find

$$\sum_{n=1}^{\infty} \frac{A_n - A_{n-k}}{A_k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} x^n = \sum_{n=1}^{\infty} \sum_{d|n} x^r R_k(d, A)$$

$$= \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} x^{md} R_k(d, A) \quad , \quad \text{by (38)} \quad ,$$

$$= \sum_{d=1}^{\infty} R_k(d, A) \sum_{m=1}^{\infty} x^{md} = \sum_{d=1}^{\infty} R_k(d, A) \frac{x^d}{1 - x^d} \quad .$$

The lower limits of summation in the result can be changed to k instead of 1 since the Fontené-Ward coefficients and R_k are each zero for the first $k - 1$ terms on each side. This proves the theorem.

We have given some detailed steps to illustrate precisely what happens. But let us now try to carry over the binomial theorem. It turns out that we do not need the binomial theorem in a very strong form.

To find the series expansion inverse to (39), we recall the bracket function series (of Hermite) from [8];

$$\sum_{n=k}^{\infty} \left[\frac{n}{k} \right] x^n = \frac{x^k}{(1 - x)(1 - x^k)} \quad , \quad k \geq 1. \tag{41}$$

Substitute the expansion of $[n/k]$ in terms of Fontené-Ward coefficients, and we get

$$\frac{x^k}{1 - x^k} = \sum_{n=k}^{\infty} (1 - x) x^n \sum_{j=k}^{n} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} Q_k(j, A) \tag{42}$$

$$= \sum_{j=k}^{\infty} Q_k(j, A)\, (1 - x) \sum_{n=j}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} x^n \quad .$$

The last inner sum is not conveniently put into closed form by a binomial theorem, but we can transform it as follows:

$$(1 - x)\sum_{n=j}^{\infty} \left\{ {n \atop j} \right\} x^n = x^j + \sum_{n=j+1}^{\infty} x^n \left(\left\{ {n \atop j} \right\} - \left\{ {n-1 \atop j} \right\} \right)$$

and we can now apply the original Fontené recurrence (6) and we recall that $A_0 = 0$ so that x^j can be counted in the sum. The result is the formula

$$(1 - x)\sum_{n=j}^{\infty} \left\{ {n \atop j} \right\} x^n = \sum_{n=j}^{\infty} x^n \left\{ {n-1 \atop j-1} \right\} \frac{A_n - A_{n-j}}{A_j} \quad . \tag{43}$$

This formula is the general counterpart of the familiar formula

$$(1 - x)\sum_{n=j}^{\infty} \binom{n}{j} x^n = \frac{x^j}{(1 - x)^j}$$

used in [7, pp. 241, 252]. The corresponding q-analog in [8, p. 407] was

$$(1 - x)\sum_{n=j}^{\infty} \left[{n \atop j} \right] x^n = x^j \prod_{i=1}^{j} (1 - xq^i) \ .$$

The reader may find it interesting to find the corresponding Fibonomial form.

Finally, we substitute expansion (43) into (42) and we find the formula inverse to (39); i.e., we have proved

<u>Theorem 5.</u> The coefficients $Q_k(j, A)$ satisfy the generating expansion

$$\frac{x^k}{1 - x^k} = \sum_{j=k}^{\infty} Q_k(j, A) \sum_{n=j}^{\infty} x^n \left\{ {n-1 \atop j-1} \right\} \frac{A_n - A_{n-j}}{A_j} \tag{44}$$

We may write the two expansions of Theorems 4 and 5 in the forms

$$\sum_{j=k}^{\infty} R_k(j, A) \frac{x^j}{1 - x^j} = f(x, k) , \tag{45}$$

and

$$\sum_{j=k}^{\infty} Q_k(j, A) f(x, j) = \frac{x^k}{1 - x^k} , \tag{46}$$

where $f(x, j)$ is the power series

$$f(x, j) = \sum_{n=j}^{\infty} x^n \left\{ \begin{matrix} n-1 \\ j-1 \end{matrix} \right\} \frac{A_n - A_{n-j}}{A_j} = (1 - x) \sum_{n=j}^{\infty} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} x^n , \tag{47}$$

and we may now see easily that substitution of (45) into (46), and conversely, yields our desired orthogonality of R and Q. Thus we evidently have

Theorem 6. The functions R and Q as defined by (20) and (37) satisfy the orthogonality relations

$$\sum_{j=k}^{n} R_k(j, A) Q_j(n, A) = \delta_k^n = \sum_{j=k}^{n} Q_k(j, A) R_j(n, A) . \tag{48}$$

Consequently, we also have proved the very general inversion theorem for two sequences that held for the previous cases [7], [8]. That is we have

Theorem 7. For two sequences $f(x, k, A)$, $g(x, k, A)$, then

$$f(x, k, A) = \sum_{k \le j \le x} g(x, j, A) R_k(j, A) \tag{49}$$

if and only if

$$g(x,k,A) = \sum_{k \le j \le x} f(x,j,A) Q_k(j,A) . \tag{50}$$

CONCLUSION

In the present paper we have given a sequence of seven main theorems, generalizing all of the corresponding results previously found for ordinary and q-binomial coefficients to the most general situation for Fontené-Ward generalized binomial coefficients. As a single byproduct we have results universally valid for the popular Fibonomial triangle. The inversion theorems given here are expected to suggest other inversion theorems in the most general setting, which can then be applied to any special case that is covered by the Fontené-Ward Triangle.

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