OVERLAYS OF PASCAL'S TRIANGLE

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The purpose of this paper is to demonstrate the versatility of the method presented by V. E. Hoggatt, Jr. It is hoped that the examples presented in this paper will demonstrate to the reader some of the research possibilities opened by this method. (See [1].)

THE METHOD

The basis of the method lies in the concept of generating functions for the columns of a left-adjusted Pascal's triangle. From Figure 1, we see that the generating function for the k^{th} column is

$$\frac{x^k}{(1-x)^{k+1}} \quad .$$

Extensive use will be made of these generating functions and certain variations of them.

row 0	1					
row 1	1	1				
row 2	1	2	1			
row 3	1	3	3	1		
row 4	1	4	6	4	1	
	⋮	⋮	⋮	⋮	⋱	
column	0	1	2	3	4	...
generating function	$\frac{1}{1-x}$	$\frac{x}{(1-x)^2}$	$\frac{x^2}{(1-x)^3}$	$\frac{x^3}{(1-x)^4}$	$\frac{x^4}{(1-x)^5}$	...

Fig. 1 Left-Adjusted Pascal's Triangle

ARRAY OVERLAYS

Consider the two arrays A and B with integer entries:

$$
\begin{array}{ccccc}
 & & & & \cdot^{\cdot^{\cdot}} \\
 & & & a_{44} & \cdots \\
 & & a_{33} & a_{34} & \cdots \\
 & a_{22} & a_{23} & a_{24} & \cdots \\
a_{11} & a_{12} & a_{13} & a_{14} & \cdots
\end{array}
$$

$$
\begin{array}{ccccc}
b_{11} & & & & \\
b_{12} & b_{22} & & & \\
b_{13} & b_{23} & b_{33} & & \\
b_{14} & b_{24} & b_{34} & b_{44} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

An overlay of A on B means that a sequence $C = \{c_1, c_2, \cdots\}$ is produced such that:

$$
\begin{aligned}
c_1 &= a_{11} \cdot b_{11} \\
c_2 &= a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\
c_3 &= a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} + a_{22} \cdot b_{22} \\
&\ \vdots \\
c_i &= \sum_{k=0}^{\left[\frac{i}{2}\right]} \sum_{M=k+1}^{i-k} a_{k+1\,M} b_{M\,i-k} , \\
&\ \vdots
\end{aligned}
$$

where $[s]$ as usual represents the greatest integer in s.

FOUR EXAMPLES

Example I. Let us see what type of sequence we can expect if A and B are both left-adjusted Pascal's triangles (i. e., A is a left-adjusted Pascal's triangle placed on its side). The first few terms of such an overlay are

$$1, \quad 2, \quad 5, \quad 12, \quad 29, \quad \cdots$$

which suggests that there is a recursive relationship that is described by the rule

$$U_{n+2} = 2U_{n+1} + U_n \; .$$

The verification that this recursion indeed holds for the whole sequence can be accomplished by noting that the coefficients of the expansion of $(1 + x)^n$ represent the n^{th} row of Pascal's triangle and that in the overlay the n^{th} row of Pascal's triangle lies on the n^{th} column. Hence we arrive at the conclusion that the generating function for the sequence is

$$\frac{1}{1-x} + (1+x)\frac{x}{(1-x)^2} + (1+x)^2\frac{x^2}{(1-x)^3} + \cdots + (1+x)^{n-1}\frac{x^{n-1}}{(1-x)^n} + \cdots$$
$$= \frac{1}{1-x}\left\{\frac{1}{1 - \frac{x(1+x)}{1-x}}\right\} = \frac{1}{1-2x-x^2} \; .$$

The reader can easily verify that

$$\frac{1}{1 - 2x - x^2}$$

generates a sequence where the desired recursive relation holds. This example shows that, in spite of the seemingly formidable configuration of the elements of the sequence C, with the column generators one is able to cope with the situation easily.

Example II: This example will concern itself with determining which arrays, when overlayed, will yield the Fibonacci sequence. In order to effect this, we will begin with the generating function for the Fibonacci sequence which is

$$\frac{1}{1-x-x^2} = \frac{1+x}{1-x^2}\left\{\frac{1}{1-\dfrac{x^2(1+x)}{1-x^2}}\right\}$$

$$= \frac{1+x}{1-x^2} + \frac{x^2(1+x)^2}{(1-x^2)^2} + \cdots + \frac{x^{2(n-1)}(1+x)^n}{(1-x^2)^n} + \cdots \;.$$

Remembering Example I, the presence of $(1+x)^{n-1}$ in the n^{th} term suggests that the A array is a left-adjusted Pascal's triangle. Then the B array must have column generators of

$$\frac{1+x}{1-x^2},\; \frac{x^2(1+x)}{(1-x^2)^2},\; \frac{x^4(1+x)}{(1-x^2)^3},\; \cdots,\; \frac{x^{2(n-1)}(1+x)}{(1-x^2)^n} \;.$$

If one notes that x^2 has replaced the x in Figure 1 and that the $1+x$ "fills in" the void left by that replacement, then the array with these column generators is easily seen to be

$$\begin{array}{ccccc}
1 \\
1 \\
1 & 1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 3 & 3 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

which is a doubled left-adjusted Pascal's triangle. (Note that, for example, the spot a_{22} is not listed. Consider those spots to contain zero.)

Therefore we conclude that the Fibonacci sequence can be generated by overlaying the left-adjusted Pascal's triangle on the doubled left-adjusted Pascal's triangle.

Example III: In this example the results from Example II will be carried one more step toward a generalization. Instead of considering the Fibonacci sequence, a Fibonacci-like sequence will be considered,

$$U_1 = 1, \quad U_2 = 1, \quad U_3 = 1, \quad U_{n+3} = U_{n+2} + U_{n+1} + U_n ,$$

The generating function for this sequence is easily found, see [2], to be

$$\frac{1 - x^2}{1 - x - x^2 - x^3} = 1 + \frac{x(1+x^2)}{1-x^2} + \frac{x^2(1+x^2)^2}{(1-x^2)^2} + \cdots + \frac{x^n(1+x^2)^n}{(1-x^2)^n} .$$

The presence of $(1 + x^2)^{n-1}$ in the numerator of the n^{th} term suggests that the A array is

$$\begin{array}{ccccl}
 & & & & \cdot^{\cdot^{\cdot}} \\
 & & & 1 & \cdots \\
 & & & 0 & \cdots \\
 & & 1 & 3 & \cdots \\
 & & 0 & 0 & \cdots \\
 & 1 & 2 & 3 & \cdots \\
 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & \cdots
\end{array}$$

which is simply a left-adjusted Pascal's triangle with a column of zeros placed in between each of its columns. Note that this array is not in the exact form of the array A but the analogous method of overlaying this array is obvious. We are now left with the generating functions

$$1, \quad \frac{x}{1 - x^2}, \quad \frac{x^2}{(1 - x^2)^2}, \cdots$$

which yield the array

$$\begin{array}{cccccccc}
1 \\
0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 & 1 \\
0 & 1 & 0 & 3 & 0 & 1 \\
0 & 0 & 3 & 0 & 4 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Therefore, using the method of column generators, we have found the proper arrays which overlay to form the given sequence.

Example IV: Consider the generalized Pascal's triangle whose k^{th} row is determined by the coefficients of the expansion of

$$(1 + x + \cdots + x^{r-1})^k; \quad k = 0, 1, \cdots \text{ and } r \geq 2.$$

Let this triangle be the A-array and let

$$\begin{array}{cccc}
1 \\
1 \\
1 & 1 \\
1 & 2 \\
1 & 3 & 1 \\
1 & 4 & 3 & \ddots \\
\vdots & \vdots & \vdots
\end{array}$$

be the B-array. Note that the B-array is formed by "pushing" the columns of Pascal's triangle down so that the first entry of the k^{th} column appears in the $2k^{th}$ row; $k = 0, 1, 2, \cdots$. Hence by our prior experience we know that the generator for the k^{th} column of the B-array is

$$\frac{x^{2k}}{(1 - x)^{k+1}} \quad .$$

By the method used in the previous examples, the generator for the sequence determined by overlaying A on B is

$$\frac{1}{(1 - x)} + (1 + x + \cdots + x^{r-1})\left(\frac{x^2}{(1 - x)^2}\right) + (1 + \cdots + x^{r-1})^2\left(\frac{x^4}{(1 - x)^3}\right) + \cdots$$

$$= \frac{1}{1 - x - x^2 - \cdots - x^{r+1}} \quad .$$

It is easy to verify that

$$\frac{1}{1 - x - x^2 - \cdots - x^{r+1}} = u_1 + u_2 x + u_3 x^2 + \cdots$$

where

$$u_1 = 1, \quad u_2 = 1, \quad u_3 = 2, \quad \cdots, \quad u_{r+1} = 2^{r+1}, \quad u_{r+2} = 2^r ,$$

$$u_n = \sum_{i=1}^{r+1} u_{n-i}$$

for $n > r + 2$.

It is interesting to note that this sequence of u's is precisely the sequence of the rising diagonal sums in the generalized Pascal's triangle whose k^{th} row is determined by the coefficients of the expansion of $(1 + x + \cdots + x^{r+1})^k$; $k = 0, 1, 2, \cdots$. See [2] for the proof of this fact and [3] for a further discussion of related subjects.

CONCLUSION

The approach used in the preceding examples to find the sequence determined by overlaying an array A on an array B can be described as follows. Let $P_k(x) = a_{1k} + a_{2k}x + \cdots + a_{kk}x^{k-1}$, and let $G_k(x)$ be the generating function for the k^{th} column of the B array; $k = 0, 1, 2, \cdots$. Then

$$\sum_{i=0}^{\infty} P_i(x) G_i(x)$$

is the generating function that determines the desired sequence.

Almost an unlimited number of problems of the type worked in this paper are now open to scrutiny. At the end of this paper there are two such problems stated. The first one is fairly straight forward and the ultimate answer is supplied. The second one seems to be a little tougher and might make a nice project for some ambitious student.

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PROBLEMS

Problem I: Let an unending row of urns be given, the first one labeled 0, the second labeled 1 and so forth. In the urn labeled "k" let there be k distinguishable balls; $k = 0, 1, 2, \cdots$. Suppose a man does a series of events with the n^{th} event, $n = 0, 1, 2, \cdots$, described as follows:

a) He reaches into the urn labeled "n" $n + 1$ times. The first time he takes out 0 balls, the second time 1 ball, the third 2 balls and so forth until the $n + 1$ time he removes all n balls each time replacing the balls he has previously removed.

b) In general he reaches into the urn marked "$n - j$" $n - 2j$ times taking out $j, j + 1, \cdots, n - j$ balls respectively (again by replacement), $j \geq 0$.

c) This event ends when he has moved down the line of urns to the one labeled $n - s$ such that $n - s < s$ for the first time.

Since the balls are distinct, associated with each extraction of balls (i.e., each time the man reaches into an urn) there is a number which represents the number of ways the extraction could have occurred. Let S_k be the sum of all these numbers in the k^{th} event. The problem is to find a generating function that determines $\{S_i\}_{i=0}^{\infty}$ as its sequence.

Ans. $$\left(\frac{1}{1 - 3x + 2x^2} - \frac{x}{1 - 2x + x^3}\right)$$

Problem II. Find two non-trivial arrays such that their overlay determines the sequence:

$$U_0 = U_1 = \cdots = U_{n-1} = 1 \quad \text{and} \quad U_k = U_{k-1} + \cdots + U_{k-n}$$

for all $k \geq n$. See [4].

REFERENCES

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4. Problem H-87, Fibonacci Quarterly, Vol. 4, No. 2, April, 1966, page 149, by M. B. Boisen, Jr.

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