SUMMATION OF INFINITE FIBONACCI SERIES

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In a previous paper, a well-known technique for summing finite or infinite series was employed to arrive at a number of summations of Fibonacci and Lucas infinite series in closed form [1]. This work is rewarding but in reality covers only a limited portion of the possible infinite series that can be constructed. Starting in general with an arbitrary Fibonacci or Lucas infinite series, the probability that it has a closed sum is relatively small. One need only think of the sum of the reciprocals of the Fibonacci numbers themselves which to date has not been determined in a precise manner.

In the face of this situation, what remains to be done? The present article attacks this problem by attempting to accomplish two things: (1) Determining the relations among cognate formulas so that formulas can be grouped into families in which all the members of one family are expressible in terms of one member of the family and other known quantities; (2) Replacing slowly converging sums by those that converge more rapidly.

The combination of these two efforts has this effect. Given families A_1, A_2, $A_3, \cdots$, whose members are expressible in terms of summations a_1, a_2, $a_3, \cdots$, respectively. Then if these quantities a_i can be related to other quantities a_i' which converge more rapidly, the problem of finding the summations in the various families is reduced once and for all to making precise determinations of a very few summations a_i' which can be found in a reasonably small number of steps.

Such is the program. The purpose of the article is to give an illustrative rather than an exhaustive treatment. The investigation, moreover, will be limited to infinite series of the type:

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k_1} F_{n+k_2} F_{n+k_3} \cdots F_{n+k_r}}$$

or of the form:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_r}}$$

with all the Fibonacci numbers in the denominator different and the k_i positive.

NOTATION AND LANGUAGE

To compress notation, the expression $(F_n)_r$ will mean

$$F_n F_{n-1} F_{n-2} \cdots F_{n-r+1} \quad .$$

If there are k Fibonacci numbers in a denominator, we shall speak of this as a "summation of the k^{th} degree."

CONVERGENCE OF THE SUM OF FIBONACCI RECIPROCALS

We shall begin by establishing the fact that the sum of the reciprocals of the Fibonacci numbers:

$$\sum_{n=1}^{\infty} 1/F_n$$

converges. This in turn will be sufficient in itself to enable us to conclude to the convergence of all sums of our two types since their terms are less than or equal to those of this series.

Using the roots of the equation $x^2 - x - 1 = 0$, namely,

$$r = \frac{1+\sqrt{5}}{2} \text{ and } s = \frac{1-\sqrt{5}}{2} \quad ,$$

we have

$$F_n = \frac{r^n - s^n}{\sqrt{5}} \quad . \tag{1}$$

Now $s = -r^{-1}$. Hence when n is odd,

$$1/F_n < \sqrt{5}/r^n$$

and when n is even, it can be shown that

$$1/F_n < \sqrt{5}/r^{n-1} .$$

This follows since the relation for n even, $r^n - r^{-n} > r^{n-1}$ leads to $r^n - r^{n-1} > r^{-n}$, or finally $r - 1 > r^{-2n+1}$ which is certainly true for $n \geq 2$; for $n = 1$, $r - 1 = r^{-1}$.

Thus in either case

$$1/F_n < \sqrt{5}/r^{n-1}, \text{ for } n \geq 2 .$$

Hence

$$\sum_{n=1}^{\infty} 1/F_n < \sum_{n=1}^{\infty} \sqrt{5}/r^{n-1} = \frac{\sqrt{5}}{1 - 1/r} \tag{2}$$

Since the summation of positive terms has an upper bound, it follows that it must converge.

RELATIONS AMONG SECOND-DEGREE SERIES

Essentially, there is only one first degree series of each type in the sense defined in this treatment, so that the first opportunity to relate series comes with the second degree. Here we have a special situation inasmuch as the alternating series can all be evaluated, the final result being:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k}} = \frac{1}{F_k}\left[kr^{-1} - \sum_{j=1}^{k} F_{j-1}/F_j\right] \tag{3}$$

r being defined as before. The proof is as follows.

$$\sum_{n=1}^{\infty}\left[F_{n-1}/F_n - F_{n+k-1}/F_{n+k}\right] =$$

$$= \lim_{n\to\infty}\left[\sum_{j=1}^{k} F_{j-1}/F_j - \sum_{m=n-k+1}^{n} F_{n+k-1}/F_{m+k}\right] =$$

$$= \sum_{j=1}^{k} F_{j-1}/F_j - k\,r^{-1} .$$

But the initially given summation also equals

$$\sum_{n=1}^{\infty}\left[\frac{F_{n-1}F_{n+k} - F_n F_{n+k-1}}{F_n F_{n+k}}\right]$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n F_k}{F_n F_{n+k}} .$$

Equating the two values and solving gives relation (3).

The non-alternating series of the second degree has closed formulas for the summation

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+k}}$$

when k is even. For the case $k = 2$,

$$\sum_{n=1}^{\infty}\left[\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1} F_{n+2}}\right] = 1 .$$

But this likewise equals

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}}$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1 \tag{4}$$

For $k = 4$, the derivation is as follows.

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} - 3\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+4}} =$$

$$= \sum_{n=1}^{\infty} \frac{F_{n+4} - 3F_{n+2}}{F_n F_{n+2} F_{n+4}} = \sum_{n=1}^{\infty} \frac{F_n}{F_n F_{n+2} F_{n+4}}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{F_{n+2} F_{n+4}}$$

$$= \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} - \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} .$$

Solving for the desired summation,

$$3\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+4}} = 2\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} - 5/6$$

so that

(5)
$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+4}} = 2/3 - 5/18 = 7/18 .$$

The process can be contained yielding:

(6)
$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+6}} = 143/960$$

and an endless series of formulas with a closed value.

For k odd

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 2\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+3}} = \sum_{n=1}^{\infty} \frac{1}{F_{n+1} F_{n+3}}$$

$$= \frac{-1}{1 \cdot 2} + \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = -1/2 + 1 = 1/2 .$$

Therefore

(7)
$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+3}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 1/4 .$$

It is possible to proceed step-by-step to other formulas in the series. Thus

$$2\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+3}} - 5\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+5}} = \sum_{n=1}^{\infty} \frac{1}{F_{n+3} F_{n+5}}$$

$$= -\frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 5} + \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} .$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+5}} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 1/10 + 1/5(1/2 + 1/3 + 1/10 - 1)$$

or

(8) $$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+5}} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - 17/150 .$$

In summary, for second-degree summations of the given types, apart from the results in closed form, the summations

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}}$$

with k odd are all expressible in the form

$$a + b\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}}$$

where a and b are rational numbers.

AUXILIARY TABLE

In the work with these summations, the formula that is being employed to arrive at Fibonacci numbers which are to be eliminated is:

(9) $$F_k F_{k+n+r} - F_{k+r} F_{k+n} = (-1)^{k-1} F_r F_n$$

or

(10) $$F_n = \frac{(-1)^{k-1}}{F_r}\left[F_k F_{k+n+r} - F_{k+r} F_{k+n}\right]$$

Rather than use this for each instance it is found to be more convenient to make a table which indicates factors a and b in the relation:

$$F_n = a\,F_{n+k} + b\,F_{n+j} \tag{11}$$

The quantities F_{n+k} are at the right; the quantities F_{n+j} are at the top. The tabular values for any given pair are a, b in sequence. Thus, to express F_n in terms of F_{n+6} and F_{n+2}, the quantities a and b are -1/3 and 8/3, respectively, so that

$$F_n = (-F_{n+6} + 8F_{n+2})/3$$

Similarly, to express F_{n+3} in terms of F_{n+8} and F_{n+5}, since the shift in subscripts is relative, we take the table values for F_{n+5} and F_{n+2}. Hence

$$F_{n+3} = (-F_{n+8} + 5F_{n+5})/2$$

Table I
QUANTITIES a,b IN FORMULA (11)

F_{n+k} \ F_{n+j}	F_{n+1}	F_{n+2}	F_{n+3}	F_{n+4}	F_{n+5}	F_{n+6}
F_{n+2}	1, -1					
F_{n+3}	1. -2	-1, 2				
F_{n+4}	1/2(1, -3)	-1, 3	2, -3			
F_{n+5}	1/3(1, -5)	1/2(-1, 5)	2, -5	-3, 5		
F_{n+6}	1/5(1, -8)	1/3(-1, 8)	1/2(2, -8)	-3, 8	5, -8	
F_{n+7}	1/8(1, -13)	1/5(01, 13)	1/3(2, -13)	1/2(-3, 13)	5, -13	-8, 13

THIRD-DEGREE SUMMATIONS

For third-degree summations

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a} F_{n+b}}$$

there is one which has its sum in closed form. The derivation follows.

$$\sum_{n=1}^{\infty}\left[\frac{1}{F_n F_{n+1} F_{n+2}} - \frac{1}{F_{n+1}F_{n+2}F_{n+3}}\right] = \frac{1}{2} .$$

But this also equals

$$\sum_{n=1}^{\infty}\left[\frac{F_{n+3} - F_n}{(F_{n+3})_4}\right] = 2\sum_{n=1}^{\infty}\frac{F_{n+1}}{(F_{n+3})_4}$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2}F_{n+3}} = \frac{1}{4} . \tag{12}$$

To find

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+4}}$$

in terms of

$$\sum_{n=1}^{\infty}\frac{1}{(F_{n+2})_3} ,$$

we use this result, arranging coefficients so that we obtain F_n in the numerator and then eliminate it from the denominator. Thus

$$2\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+3}} - 3\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+4}}$$

$$\sum_{n=1}^{\infty} \frac{2\,F_{n+4} - 3\,F_{n+3}}{F_n\,F_{n+2}\,F_{n+3}\,F_{n+4}} = \sum_{n=1}^{\infty} \frac{F_n}{F_n\,F_{n+2}\,F_{n+3}\,F_{n+4}} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_3} = \frac{1}{1\cdot 1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3} \ .$$

Solving for the desired summation,

$$\sum_{n=1}^{\infty} \frac{1}{F_n\,F_{n+2}\,F_{n+4}} = \frac{7}{18} - \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3} \ . \tag{13}$$

The procedure is similar at each step. There are two formulas (a) and (b) to be combined with appropriate coefficients; a certain F_r is eliminated; a formula (c) is obtained, either

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3}$$

or one which has previously been expressed in terms of this quantity.

It would occupy altogether too much space to present even a small portion of the derivations. The sequence of steps, however, can be indicated by giving the denominators in the summations (a), (b), and (c) and between (b) and (c), the quantity F_r which was eliminated. The denominator of the desired summation is the same as (b) in this table.

Table II
SCHEMATIC SEQUENCE FOR THIRD-DEGREE SUMMATIONS

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a} F_{n+b}}$$

Denominator (a)	Denominator (b)	F_r	Denominator (c)
$F_n F_{n+2} F_{n+4}$	$F_n F_{n+2} F_{n+5}$	F_n	$F_{n+2} F_{n+4} F_{n+5}$
$F_n F_{n+2} F_{n+3}$	$F_n F_{n+3} F_{n+4}$	F_n	$F_{n+2} F_{n+3} F_{n+4}$
$F_n F_{n+3} F_{n+4}$	$F_n F_{n+3} F_{n+5}$	F_n	$F_{n+3} F_{n+4} F_{n+5}$
$F_n F_{n+1} F_{n+2}$	$F_n F_{n+1} F_{n+3}$	F_n	$F_{n+1} F_{n+2} F_{n+3}$
$F_n F_{n+1} F_{n+3}$	$F_n F_{n+1} F_{n+4}$	F_n	$F_{n+1} F_{n+3} F_{n+4}$
$F_n F_{n+1} F_{n+4}$	$F_n F_{n+1} F_{n+5}$	F_n	$F_{n+1} F_{n+4} F_{n+5}$
$F_n F_{n+2} F_{n+5}$	$F_n F_{n+2} F_{n+6}$	F_n	$F_{n+2} F_{n+5} F_{n+6}$
$F_n F_{n+3} F_{n+4}$	$F_n F_{n+4} F_{n+5}$	F_n	$F_{n+3} F_{n+4} F_{n+5}$
$F_n F_{n+3} F_{n+4}$	$F_n F_{n+4} F_{n+6}$	F_n	$F_{n+3} F_{n+4} F_{n+6}$

and so on.

The results can be summarized in the form

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+s} F_{n+t}} = c + dS \tag{14}$$

where

$$S = \sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3} \quad .$$

Table III
CONSTANTS c AND d FOR GIVEN s AND t IN FORMULA (14)

s,t	c	d
1,3	-1/4	1
1,4	-7/36	2/3
1,5	-71/450	7/15
1,6	-509/4800	3/10
1,7	-11417/162240	5/26
2,3	1/4	0
2,4	7/18	-1/3
2,5	71/300	-1/5
2,6	509/2880	-1/6
2,7	11417/101400	-7/65
3,4	-5/36	1/3
3,5	-67/300	2/5
3,6	-269/1920	1/4
4,5	19/225	-1/15
4,6	407/2880	-1/6

It should be apparent without formal proof that any summation of the third degree with positive terms can be expressed in the form given by (14). A practical conclusion follows: It is only necessary to find the value of one summation

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+2})_3}$$

which can be done once and for all to any desired number of decimal places. Thereafter for formulas related to this summation their values can be found with a minimum of effort to any desired number of places within the limits established for the one basic formula.

This method of relating a number of formulas to one formula can be continued to higher degrees though the complexities become greater. For example, for seventh-degree expressions:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=1}^{7} F_{n+k_i}}$$

we can proceed step-by-step according to the following table. The quantities (a), (b), (c) and F_r have the same meaning as in Table II. The desired summation is indicated by an asterisk.

Table IV
SCHEMATIC SEQUENCE FOR SEVENTH-DEGREE SUMMATIONS WITH ALTERNATING TERMS

Denominator (a)	Denominator (b)	F_r	Denominator (c)
$(F_{n+6})_7$	$^*(F_{n+5})_6 F_{n+7}$	F_n	$(F_{n+7})_7$
$(F_{n+6})_7$	$^*(F_{n+5})_6 F_{n+8}$	F_n	$(F_{n+6})_6 F_{n+8}$
$(F_{n+6})_7$	$^*(F_{n+5})_6 F_{n+9}$	F_n	$(F_{n+6})_6 F_{n+9}$
and so on.			
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+7}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+6} F_{n+7}$
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+8}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+6} F_{n+8}$
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+9}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+6} F_{n+9}$
and so on.			
$(F_{n+5})_6 F_{n+7}$	$(F_{n+5})_6 F_{n+8}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+7} F_{n+8}$
$(F_{n+5})_6 F_{n+7}$	$(F_{n+5})_6 F_{n+9}$	F_{n+5}	$^*(F_{n+4})_5 F_{n+7} F_{n+9}$
and so on.			
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+7}$	F_{n+4}	$^*(F_{n+3})_4 (F_{n+7})_3$
$(F_{n+6})_7$	$(F_{n+5})_6 F_{n+8}$	F_{n+4}	$^*(F_{n+3})_4 F_{n+5} F_{n+6} F_{n+8}$
and so on.			

GENERAL CONCLUSION

It is possible to express all summations

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{i=1}^{r} F_{n+k_i}}$$

in the form

$$a + b \sum_{n=1}^{\infty} \frac{1}{(F_{n+r-1})_r} ,$$

where a and b are rational numbers; and all summations

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=1}^{r} F_{n+k_i}}$$

in the form

$$c + d \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+r-1})_r} ,$$

where again c and d are rational numbers.

The limitation of this approach is that it is not possible to proceed directly in one step to this final result as a rule. It is necessary to go through a series of formulas and should the desired summation be remote from the final objective, this could be a long operation. Once, however, the various formulas have been linked to the one formula, the problem of calculating these summations becomes relatively simple.

This concludes the discussion of linking formulas of the same degree. We now proceed to a consideration of expressing a summation of lower degree in terms of one of higher degree so as to secure more rapid convergence. But first, formulas will be worked out giving upper bounds for the number of terms required to secure a summation result correct to a given number of decimal places.

APPROXIMATING SUMMATIONS WITH GIVEN ACCURACY

Assuming that we have related the summations of a given degree to one summation, it is only necessary to consider summations of the forms

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q} \quad .$$

Two cases will be taken up according as q is even or odd.

<u>q even</u>

From previous discussion,

$$1/F_n < \sqrt{5}/r^n \quad \text{if } n \text{ is odd}$$

$$1/F_n < \sqrt{5}/r^{n-1} \quad \text{if } n \text{ is even.}$$

For

$$\frac{1}{(F_{n+q-1})_q} \quad ,$$

the result depends on the power of r found on the right-hand side of the inequality. These powers can be calculated by table as follows.

n odd	n even
$2n$	$n - 1$
$2(n + 2)$	$2(n + 1)$
$2(n + 4)$	$2(n + 3)$
$\cdots$	$\cdots$
$2(n + q - 2)$	$2(n + q - 3)$
	$n + q - 1$

The sum in either case is

$$qn + \frac{q(q - 2)}{2} \quad .$$

If we want w terms of the summation

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q}$$

to give a result correct to t decimal places, the $(w + 1)^{st}$ term must be less than 5×10^{-t}. Hence the condition for the desired upper bound is:

$$\frac{1}{(F_{w+q})_q} < \frac{5^{q/2}}{r^{q(w+1)+q(q-2)/2}} < 5 \times 10^{-t}$$

which leads to

$$\frac{t + \frac{(q-2)}{2} \log 5 - \frac{q^2}{2} \log r}{q \log r} < w$$

or

$$w > \frac{4.78514}{q} t + \frac{(q-2)}{2q} (3.34467) - \frac{q}{2} \tag{15}$$

For example, if q is 8 and we want the result correct to 10 decimal places,

$$w > .59814t - 2.74575 \text{ or } w > 3.3565 .$$

Hence four terms would be required. Data from this formula will be found in Table V.

For the summation

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q}$$

to have the result correct to t decimal places,

$$\sum_{k=w+1}^{\infty} \frac{1}{(F_{k+q-1})_q} < \sum_{k=w+1}^{\infty} \frac{5^{q/2}}{r^{qk+q(q-2)/2}}$$

$$= \frac{5^{q/2}}{r^{qw+q^2/2}} \left[1 + r^{-q} + r^{-2q} \cdots\right] < 5 \times 10^{-t}$$

or

$$\frac{5^{q/2}}{r^{qw+q^2/2}} \frac{1}{1 - r^{-q}} < 5 \times 10^{-t}$$

This leads to the inequality

$$w > \frac{t + \frac{(q-2)}{2} \log 5 - \log (1 - r^{-q}) - \frac{q^2 \log r}{2}}{q \log r} .$$

The term

$$-\log (1 - r^{-q}) = r^{-q} + \frac{r^{-2q}}{2} + \frac{r^{-3q}}{3} \cdots$$

$$< r^{-q} + r^{-2q} + r^{-3q} \cdots = \frac{1}{r^q - 1} .$$

This replacement is in the safe direction. Hence

(16) $$w > \frac{4.78514t}{q} + \frac{3.34467(q-2)}{2q} - \frac{q}{2} + \frac{1}{(r^q - 1) q \log r} .$$

Similar considerations applied to the case of q odd give the following results. For the summation with alternating terms:

(17) $$w > \frac{4.78514t}{q} + \frac{3.34467\ (q-2)}{2q} - \frac{(q^2 - 1)}{2q} .$$

For the summation with all terms positive:

$$w > \frac{4.78514t}{q} + \frac{3.34467(q-2)}{2q} + \frac{4.78514}{q(r^q - 1)} - \frac{(q^2-1)}{2q} \quad . \tag{18}$$

Table V

UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q}$$

t \ q →	2	4	6	8	10	12	14	16	18	20
5	11	5	3	1						
10	23	11	7	4	2					
15	35	17	11	7	4	2				
20	47	23	15	10	6	4	2			
25	59	29	19	13	9	6	3	1		
30	71	35	23	16	11	8	5	3	1	
50	119	59	38	28	21	16	12	9	6	4
100	239	119	78	58	45	36	29	24	20	16

Table VI

UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q}$$

t \ q	2	4	6	8	10	12	14	16	18	20
5	13	6	3	1						
10	25	12	7	4	2					
15	37	17	11	7	4	2				
20	49	23	15	10	6	4	2			
25	61	29	19	13	9	6	3	1		
30	73	35	23	16	11	8	5	3	1	
50	121	59	39	28	21	16	12	9	6	4
100	240	119	78	58	45	36	29	24	20	16

Table VII
UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+q-1})_q}$$

t \ q	1	3	5	7	9	11	13	15	17	19
5	23	8	4	2						
10	47	16	9	5	3	1				
15	71	24	13	9	5	3	1			
20	95	32	18	12	8	5	3	1		
25	119	40	23	15	11	7	5	2	1	
30	143	48	28	19	13	9	6	4	2	
50	239	79	47	32	24	18	14	10	8	5
100	478	159	95	67	51	40	32	26	22	18

Table VIII
UPPER BOUNDS FOR THE NUMBER OF TERMS REQUIRED FOR RESULTS TO t DECIMAL PLACES FOR THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+q-1})_q}$$

t	1	3	5	7	9	11	13	15	17	19
5	31	8	4	2						
10	55	16	9	5	3	1				
15	79	24	14	9	5	3	1			
20	103	32	18	12	8	5	3	1		
25	127	40	23	15	11	7	5	2	1	
30	151	48	28	19	13	9	6	4	2	
50	246	80	47	32	24	18	14	10	8	5
100	486	160	95	67	51	40	32	26	22	18

These tables indicate impressively the gain in efficiency obtained by expressing lower-degree summations in terms of a higher degree summation. The method of achieving this will now be taken up.

LOWER DEGREE SUMMATIONS
IN TERMS OF HIGHER DEGREE SUMMATIONS

The program to be carried out illustrating this process will consist in starting with

$$\sum_{n=1}^{\infty} 1/F_n$$

and establishing a chain of formulas reaching to

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} .$$

The first step in this chain is found in a result given in the Fibonacci Quarterly [2], namely:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+2})_3} .$$

The next step is as follows.

$$(19) \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+2})_3} - 3\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{n+1} F_{n+2} F_{n+4}}$$

$$= -\frac{1}{1 \cdot 1 \cdot 3} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+3}}$$

$$= -1/3 + 1/4 = -1/12 .$$

It is possible to express

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}}$$

in terms of

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} .$$

Starting with

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{F_n F_{n+1} F_{n+3}} + \frac{1}{F_{n+1} F_{n+2} F_{n+4}} \right] = \frac{1}{1 \cdot 1 \cdot 3}$$

and noting that

$$F_{n+2} F_{n+4} + F_n F_{n+3} = (-1)^{n-1} + 2F_{n+2} F_{n+3}$$

$$1/3 = \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}} .$$

Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+4}} = \frac{1}{6} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} .$$

Substituting into (19),

(20) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+2})_3} = \frac{5}{12} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} .$$

The next step is as follows.

$$\sum_{n=1}^{\infty}\left[\frac{1}{(F_{n+3})_4 F_{n+5}} - \frac{1}{(F_{n+4})_4 F_{n+6}}\right] = \frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 8}$$

$$= \sum_{n=1}^{\infty}\left[\frac{F_{n+4}F_{n+6} - F_n F_{n+5}}{(F_{n+6})_7}\right]$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{(F_{n+2})_3 F_{n+5}F_{n+6}} + 3\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1}F_{n+3}F_{n+5}F_{n+6}}$$

$$+ 3\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_{n+6})_7} .$$

We shall not derive the relations for the fifth-degree summations in terms of

$$\sum_{n=1}^{\infty}\frac{1}{(F_{n+4})_5} = S ,$$

but simply state them.

$$\sum_{n=1}^{\infty}\frac{1}{(F_{n+2})_3 F_{n+5}F_{n+6}} = \frac{3S}{20} - \frac{1}{4800}$$

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1}F_{n+3}F_{n+5}F_{n+6}} = \frac{7S}{40} - \frac{29}{9600} .$$

Substitution leads to the result

(21)
$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+4})_5} = \frac{97}{2640} + \frac{40}{11} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+6})_7} .$$

The final stage is as follows.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{(F_{n+5})_6 F_{n+7}} + \frac{1}{(F_{n+6})_6 F_{n+8}} \right] = \frac{1}{1\cdot1\cdot2\cdot3\cdot5\cdot8\cdot21} .$$

The numerator of the combined terms inside the brackets is

$$F_{n+6} F_{n+8} + F_n F_{n+7} = 4 F_{n+4} F_{n+6} + 5 F_{n+4} F_{n+5} + 3(-1)^{n-1} .$$

Letting

$$A = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+3})_4 F_{n+5} F_{n+7} F_{n+8}}$$

$$B = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+3})_4 (F_{n+8})_3}$$

and

$$C = \sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} ,$$

the result can be written as

$$\frac{1}{1\cdot1\cdot2\cdot3\cdot5\cdot8\cdot21} = 4A + 5B + 3C .$$

Similarly,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{(F_{n+3})_4 (F_{n+7})_3} + \frac{1}{(F_{n+4})_4 (F_{n+8})_3} \right] = \frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 8\cdot 13\cdot 21} .$$

The numerator of the combined terms within the brackets is

$$F_{n+4} F_{n+8} + F_n F_{n+5} = -F_{n+4} F_{n+6} + 6 F_{n+4} F_{n+5} + 3(-1)^{n-1}$$

leading to:

$$\frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 8\cdot 13\cdot 21} = -A + 6B + 3C .$$

Solving with the previous relation in A, B, and C gives:

$$A = \frac{53}{1900080} - \frac{3C}{29} .$$

It can also be shown that

$$A = \frac{1}{91} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+6})_7} + \frac{73}{2981160} .$$

This enables us to arrive at the final conclusion

(22)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(F_{n+6})_7} = \frac{589}{1900080} - \frac{273}{29} \sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} .$$

Thus a chain has been extended from

$$\sum_{n=1}^{\infty} \frac{1}{F_n} \quad \text{to} \quad \sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} .$$

Connecting the initial and terminal links

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \frac{46816051}{13933920} + \frac{16380}{319} \sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9} \tag{23}$$

Another advantage of a summation such as

$$\sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9}$$

is that it lends itself readily to calculation. At each stage one factor is added to one end of the denominator and deleted from the other. Table IX shows the calculation of this summation to thirty plus decimal places. The factor applied at each stage to the result on the preceding line is shown at the left.

Table IX

CALCULATION OF $\sum_{n=1}^{\infty} \frac{1}{(F_{n+8})_9}$

Term	Factor	Term Multiplied by 10^{33}					
1		4488	97507	72103	71328	01838	684
2	1/55	81	61772	86765	52205	96397	067
3	1/89		91705	31311	97215	79734	799
4	2/144		1273	68490	44405	77496	317
5	3/233		16	39937	64520	24602	957
6	5/377			21749	83614	32687	042
7	8/610			285	24375	26986	060
8	13/987			3	75700	99139	634
9	21/1597				4940	33864	704
10	34/2584				65	00445	588
11	55/4181					85511	721
12	89/6765					1124	988
13	144/10946					14	800
14	233/17711						195
	SUM	4571	52276	20648	18372	59844	456

With the aid of this value

$$\sum_{n=1}^{\infty} \frac{1}{F_n}$$

is found to be to twenty-five decimal places

$$3.35988\ 56662\ 43177\ 55317\ 20113 \quad .$$

CONCLUSION

In this paper two types of infinite Fibonacci series have been considered. Methods have been developed for expressing series of the same degree in terms of one series of that degree. In addition a path has been indicated for proceeding from series of lower degree to those of higher degree so that more rapid convergence may be attained. These two approaches plus the development of closed formulas in a previous article should provide an open door for additional research and calculation along the lines of sums of reciprocals of Fibonacci series of various types.

REFERENCES

1. Brother Alfred Brousseau, "Fibonacci-Lucas Infinite Series Research," the Fibonacci Quarterly, Vol. 7, No. 2, pp. 211-217.
2. The Fibonacci Quarterly, Problem H-10, April 1963, p. 53; Solution to H-10, the Fibonacci Quarterly, Dec. 1963, p. 49.

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