

## LINEAR RECURSION RELATIONS – LESSON FOUR

### SECOND-ORDER LINEAR RECURSION RELATIONS

BROTHER ALFRED BROUSSEAU  
St. Mary's College, California

Given a second-order linear recursion relation

$$(1) \quad T_{n+1} = a T_n + b T_{n-1} ,$$

where  $a$  and  $b$  are real numbers and the values  $T_i$  of the sequence are real as well, there is an auxiliary equation:

$$(2) \quad x^2 - ax - b = 0 ,$$

with roots

$$(3) \quad r = \frac{a + \sqrt{a^2 + 4b}}{2}$$
$$s = \frac{a - \sqrt{a^2 + 4b}}{2} .$$

As is usual with quadratic equations, three cases may arise depending on whether

$$a^2 + 4b > 0, \quad \text{roots real and distinct;}$$
$$(4) \quad a^2 + 4b = 0, \quad \text{roots real and equal;}$$
$$a^2 + 4b < 0, \quad \text{roots complex numbers.}$$

#### CASE 1. $a^2 + 4b > 0.$

In previous lessons we have considered cases of this kind. It has been noted that the roots may be rational or irrational. There seems to be nothing to add for the moment to the discussion of these cases.

CASE 2.  $a^2 + 4b = 0$ .

The presence of multiple roots in the auxiliary equation clearly requires some modification in the previous development. If

$$x^2 - ax - b = 0$$

$$x^n - ax^{n-1} - bx^{n-2} = 0 .$$

Since the equation has a multiple root  $(a/2)$ , the derivative of this equation will have this same root. Hence

$$(5) \quad nx^{n-1} - a(n-1)x^{n-2} - b(n-2)x^{n-3} = 0$$

is satisfied by the multiple root also.

Thus the multiple root,  $r$ , satisfies the following two relations:

$$r^n = ar^{n-1} + br^{n-2}$$

(6)

$$nr^n = a(n-1)r^{n-1} + b(n-2)r^{n-2} .$$

The result is that if we formulate  $T_n$  as

$$T_n = Anr^n + Br^n$$

(7)

$$T_{n+1} = A(n+1)r^{n+1} + Br^{n+1}$$

it follows that

$$T_{n+2} = aT_{n+1} + bT_n$$

(8)

$$= A[a(n+1)r^{n+1} + bn r^n] + B[ar^{n+1} + br^n]$$

$$= A(n+2)r^{n+2} + Br^{n+2}$$

so that the form of  $T$  is maintained.

EXAMPLE

Find the expression for  $T_n$  in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 6T_n - 9T_{n-1}$$

if  $T_1 = 4$ ,  $T_2 = 7$ . Here the auxiliary equation is  $x^2 - 6x + 9 = 0$  with a double root of 3. Hence  $T_n$  has the form

$$T_n = Anx3^n + Bx3^n.$$

Using the values of  $T_1$  and  $T_2$

$$4 = Ax3 + Bx3$$

$$7 = 2Ax3^2 + Bx3^2$$

with solutions  $A = -5/9$ ,  $B = 17/9$ . Hence

$$T_n = \frac{-5nx3^n + 17x3^n}{9} = 3^{n-2} [(-5n + 17)].$$

It may be noted that for any non-zero multiple root  $r$ , the determinant of the coefficients in the set of equations for  $T_1$  and  $T_2$  is

$$\begin{vmatrix} r & r \\ 2r^2 & r^2 \end{vmatrix} = -r^3$$

which is not zero, so that these equations will always have a solution.

CASE 3.  $a^2 + 4b < 0$ .

The case of complex roots is quite similar to that of real and distinct roots as far as determining coefficients from initial value equations is con-

cerned. However, since we have specified that the terms of the sequence and the coefficients in the recursion relation are real, there will have to be a special relation between  $A$  and  $B$  in the expression for  $T_n$ :

$$T_n = A r^n + B s^n .$$

Note that  $r$  and  $s$  are complex conjugates, so that  $r^n$  and  $s^n$  are of the form  $P + Qi$  and  $P - Qi$  respectively, where  $P$  and  $Q$  are real. If  $T_n$  is to be real, then  $A$  and  $B$  must be complex conjugates as well.

#### EXAMPLE

Find the expression for  $T_n$  in terms of the roots of the auxiliary equation for the linear recursion relation

$$T_{n+1} = 3 T_n - 4 T_{n-1} ,$$

with  $T_1 = 5$ ,  $T_2 = 9$ . Here the auxiliary equation is:

$$x^2 - 3x + 4 = 0$$

with roots

$$r = \frac{3 + i\sqrt{7}}{2} , \quad s = \frac{3 - i\sqrt{7}}{2} .$$

Then

$$5 = Ar + Bs$$

$$9 = A r^2 + B s^2$$

from which one finds that

$$A = \frac{21 - 11i\sqrt{7}}{28} , \quad B = \frac{21 + 11i\sqrt{7}}{28} .$$

Accordingly,

$$T_n = \left( \frac{21 - 11i\sqrt{7}}{28} \right) r^n + \left( \frac{21 + 11i\sqrt{7}}{28} \right) s^n .$$

#### AN ANALOGUE

Because of the similarities among second-order linear recursion relations it is possible to find close analogues among them to the Fibonacci and Lucas sequences. Let us consider as an example the second-order linear recursion relation

$$T_{n+1} = 3T_n + T_{n-1} .$$

The auxiliary equation is

$$x^2 - 3x - 1 = 0$$

with roots

$$r = \frac{1 + \sqrt{13}}{2}, \quad s = \frac{1 - \sqrt{13}}{2} .$$

If the initial terms are taken as  $T_0 = 0$ ,  $T_1 = 1$ ,  $T_2 = 3$ , then

$$1 = Ar + Bs$$

$$3 = Ar^2 + Bs^2 ,$$

with resulting values  $A = 1/\sqrt{13}$  and  $B = -1/\sqrt{13}$  so that

$$T_n = \frac{r^n - s^n}{\sqrt{13}} = \frac{r^n - s^n}{r - s}$$

has precisely the same form as the expression for  $F_n$  with 13 replacing 5 under the square root sign.

If the relation  $V_n = T_{n+1} + T_{n-1}$  is used to define the corresponding "Lucas" sequence, the terms of this sequence are:

$$V_0 = 2, V_1 = 3, V_2 = 11, V_3 = 36, \dots$$

Solving for A and B from

$$3 = Ar + Bs$$

$$11 = Ar^2 + Bs^2$$

gives values of  $A = 1, B = 1$ , so that

$$V_n = r^n + s^n$$

in perfect correspondence to the expression for the Lucas sequence. As a result of this similarity, many relations in the Fibonacci-Lucas complex can be taken over (sometimes with the slight modification of replacing 5 by 13) to this pair of sequences. Thus:

$$T_{2n} = T_n V_n$$

$$T_{2n+1} = T_n^2 + T_{n+1}^2$$

$$T_{n+1} T_{n-1} - T_n^2 = (-1)^{n-1}$$

$$V_{2n} = V_n^2 + 2(-1)^{n+1}$$

$$V_n + V_{n+2} = 13T_{n+1}$$

$$V_n^2 + V_{n+1}^2 = 13(T_n^2 + T_{n+1}^2)$$

#### PROBLEMS

1. For the sequence  $T_1 = 1, T_2 = 3$ , obeying the linear recursion relation

$$T_{n+1} = 3T_n + T_{n-1}$$

show that every integer divides an infinity of members of the sequence.

2. For the corresponding "Lucas" sequence, prove that if  $m$  divides  $n$ , where  $n$  is odd, then  $V_m$  divides  $V_n$ .
3. Find the expression for the sequence  $T_1 = 2, T_2 = 5$  in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 4T_n + 4T_{n-1}.$$

4. Prove that the second-order linear recursion relation

$$T_{n+1} = 2T_n - T_{n-1}$$

defines an arithmetic progression.

5. If  $T_1 = a, T_2 = b$ , find the expression for  $T_n$  in terms of the roots of the auxiliary equation corresponding to  $T_{n+1} = 4T_n - 4T_{n-1}$ .
6. If  $T_1 = i, T_2 = 1$  and  $T_{n+1} = -T_{n-1}$ , find the general expression for  $T_n$  in terms of the roots of the auxiliary equation.
7.  $T_1 = 3, T_2 = 7, T_3 = 17, T_4 = 43, T_5 = 113, \dots$  are terms of a second-order linear recursion relation. Find this relation and express  $T_n$  in terms of the roots of the auxiliary equation.
8. For the second-order linear recursion relation  $T_{n+1} = 5T_n + T_{n-1}$  find the particular sequences analogous to the Fibonacci and Lucas sequences and express their terms as functions of the roots of the auxiliary equation.
9. For  $T_1 = 5, T_2 = 9, T_{n+1} = 3T_n - 5T_{n-1}$ , find  $T_n$  in terms of the roots of the auxiliary equation.
10. If

$$T_n = \left( \frac{-66 + 13\sqrt{33}}{33} \right) \left( \frac{5 + \sqrt{33}}{2} \right)^n + \left( \frac{-66 - 13\sqrt{33}}{33} \right) \left( \frac{5 - \sqrt{33}}{2} \right)^n$$

determine the recursion relation obeyed by  $T_n$  and find  $T_1$  and  $T_2$ .

[ See page 210 for Solutions to Problems. ]

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