FIBONACCI-LUCAS INFINITE SERIES – RESEARCH TOPIC

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It is almost an understatement to say that the Fibonacci Quarterly bristles with formulas. A review of this publication, however, reveals that there are very few that involve summations with Fibonacci or Lucas numbers in the denominator. Five problems in all seem to summarize the extent of what has been done along these lines in the Quarterly to February, 1966 (see references 1 to 9 inclusive). The purpose of this paper is to begin the process of filling in this gap by capitalizing on a well-known and favorite method in series summation and to provide an initial set of formulas which may form the groundwork for more extensive developments by other researchers.

The method to be employed may be illustrated by the case of

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad .$$

This can be written in the alternate form

$$\sum_{n=1}^{\infty} [1/n - 1/(n+1)] \, .$$

Let S_n be the sum of the first n terms of either the original series or of the corresponding n parentheses in the remodeled series. It follows that

$$S_n = (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \cdots + [1/(n-1) - 1/n] \, .$$

Intermediate terms add up to zero in pairs with the result that:

$$S_n = 1 - 1/n \, .$$

Now by definition, the sum of an infinite series is given by the limit of the partial sums, S_n, as n goes to infinity.

Hence

$$\sum_{n=1}^{\infty} 1/n(n+1) = S = \lim_{n \to \infty} S_n = 1 \, .$$

This method with some interesting variations will be employed in working out formulas which will provide in closed form the sums of various Fibonacci-Lucas series.

Case 1. (1) S_n contains two terms. (2) The terms of the revised series go to zero as n goes to infinity.

The example given above would correspond to this type. As an illustration, consider the summation:

$$\sum_{n=1}^{\infty} \left[1/F_{n+1} - 1/F_{n+2}\right] .$$

The sum of the first n parenthesis is:

$$S_n = (1/F_2 - 1/F_3) + (1/F_3 - 1/F_4) + \cdots + \left[1/F_{n+1} - 1/F_{n+2}\right]$$

or

$$S_n = 1 - 1/F_{n+2}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - 1/F_{n+2}) = 1 .$$

But

$$1/F_{n+1} - 1/F_{n+2} = \frac{F_{n+2} - F_{n+1}}{F_{n+1}\, F_{n+2}} = \frac{F_n}{F_{n+1}\, F_{n+2}} .$$

Accordingly

(1) $$\sum_{n=1}^{\infty} \frac{F_n}{F_{n+1}\, F_{n+2}} = 1 .$$

Case 2. (1) S_n contains more than two terms. (2) The terms of the revised series to to zero as n goes to infinity.

Example.

$$\sum_{n=1}^{\infty} [1/F_n - 1/F_{n+3}]$$

$$\begin{aligned} S_n &= (1/F_1 - 1/F_4) + (1/F_2 - 1/F_5) + (1/F_3 - 1/F_6) \\ &\qquad + (1/F_4 - 1/F_7) + \cdots + (1/F_n - 1/F_{n+3}) \\ S_n &= 1/F_1 + 1/F_2 + 1/F_3 - 1/F_{n+1} - 1/F_{n+2} - 1/F_{n+3} . \end{aligned}$$

Hence $$S = \lim_{n\to\infty} S_n = 1 + 1 + 1/2 = 5/2 .$$

But

$$1/F_n - 1/F_{n+3} = (F_{n+3} - F_n)/F_n\, F_{n+3} = 2F_{n+1}/F_n\, F_{n+3} .$$

Hence

(2) $$\sum_{n=1}^{\infty} \frac{F_{n+1}}{F_n\, F_{n+3}} = 5/4 .$$

Case 3. (1) S_n contains two terms. (2) The terms of the revised series approach a limit other than zero.

Example.

$$\sum_{n=1}^{\infty} [F_n/L_{n-1} - F_{n+1}/L_n]$$

$$S_n = (F_1/L_0 - F_2/L_1) + (F_2/L_1 - F_3/L_2) + \cdots + (F_n/L_{n-1} - F_{n+1}/L_n)$$

$$S_n = F_1/L_0 - F_{n+1}/L_n .$$

$$S = \lim_{n\to\infty} (1/2 - F_{n+1}/L_n) = 1/2 - \lim_{n\to\infty} \frac{F_{n+1}}{F_{n-1} + F_{n+1}}$$

$$S = 1/2 - \lim_{n\to\infty} \frac{F_{n+1}/F_{n-1}}{1 + F_{n+1}/F_{n-1}} .$$

If r be the Golden Section ratio

$$\frac{1+\sqrt{5}}{2},$$

$$\lim_{n\to\infty} F_{n+1}/F_{n-1} = r^2 .$$

Hence

$$S = 1/2 - r^2/(1 + r^2) = (1 - r^2)/2(1 + r^2).$$

On the other hand,

$$F_n/L_{n-1} - F_{n+1}/L_n = \frac{F_nL_n - F_{n+1}L_{n-1}}{L_{n-1}L_n} = \frac{(-1)^n}{L_{n-1}L_n} .$$

Therefore

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{n-1}L_n} = \frac{r^2 - 1}{2(r^2 + 1)} \doteq \frac{\sqrt{5}}{10}. \tag{3}$$

Case 4. S_n contains more than two terms. (2) The terms of the revised series approach a limit not zero.

Example.

$$\sum_{n=1}^{\infty} (F_{n-1}/F_n - F_{n+2}/F_{n+3})$$

$$\begin{aligned} S_n = {} & (F_0/F_1 - F_3/F_4) + (F_1/F_2 - F_4/F_5) + (F_2/F_3 - F_5/F_6) \\ & + (F_3/F_4 - F_6/F_7) \cdots (F_{n-3}/F_{n-2} - F_n/F_{n+1}) \\ & + (F_{n-2}/F_{n-1} - F_{n+1}/F_{n+2}) + (F_{n-1}/F_n - F_{n+2}/F_{n+3}) \end{aligned}$$

$$S_n = F_0/F_1 + F_1/F_2 + F_2/F_3 - F_n/F_{n+1} - F_{n+1}/F_{n+2} - F_{n+2}/F_{n+3}$$

$$S = \lim_{n\to\infty} S_n = 0 + 1 + 1/2 - 3r^{-1} = \frac{3r-6}{2r} = \frac{9-6r}{2} .$$

Now

$$F_{n-1}/F_n - F_{n+2}/F_{n+3} = \frac{F_{n-1}F_{n+3} - F_nF_{n+2}}{F_n F_{n+3}} = 2(-1)^n/F_nF_{n+3} .$$

Therefore

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+3}} = \frac{6r - 9}{4} . \tag{4}$$

ANOTHER FAMILY OF SUMMATIONS

Additional formulas can be developed by having sums of two terms in each parenthesis with the signs before the parentheses alternating.

Example 1.

$$S_n = \sum_{k=1}^{n} (-1)^{k-1} [1/(F_k L_{k+1}) + 1/(F_{k+1}L_{k+2})] .$$

Then

$$S_n = 1/(F_1L_2) + (-1)^{n-1} 1/(F_{n+1}L_{n+2})$$
$$S = \lim_{n\to\infty} S_n = 1/3 .$$

On the other hand,

$$1/(F_nL_{n+1}) + 1/(F_{n+1}L_{n+2}) = \frac{F_{n+1} L_{n+2} + F_n L_{n+1}}{F_n F_{n+1}L_{n+1}L_{n+2}} = \frac{L_{2n+2}}{F_n F_{n+1} L_{n+1} L_{n+2}} .$$

Accordingly

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}L_{2n+2}}{F_n F_{n+1} L_{n+1} L_{n+2}} = 1/3 \tag{5}$$

Example 2.

$$\sum_{n=r+2}^{\infty} (-1)^{n-1} [1/F^2_{n+r} + 1/F^2_{n-r-1}]$$

$$S_n = (1/F^2_{2r+2} + 1/F_1^2) - (1/F^2_{2r+3} + 1/F_2^2) + \cdots + (-1)^{n-1}(1/F^2_{n+2r-1} + 1/F^2_n)$$

$$S_n = \sum_{j=1}^{2r+1} (-1)^{j-1} \Big/ F_j^2 + \sum_{j=n+1}^{n+2r-1} (-1)^j \Big/ F_j^2$$

$$S = \lim_{n\to\infty} S_n = \sum_{j=1}^{2r+1} (-1)^{j-1} \Big/ F_j^2 .$$

But

$$1/F^2_{n+r} + 1/F^2_{n-r-1} = \frac{F^2_{n-r-1} + F^2_{n+r}}{F^2_{n+r}\,F^2_{n-r-1}} = \frac{F_{2r+1}\,F_{2n-1}}{F^2_{n+r}\,F^2_{n-r-1}}$$

Allowing for the fact that F_{2r+1} is a constant factor in all terms, it then follows that:

$$\text{(6)}\qquad \sum_{n=r+2}^{\infty} \frac{(-1)^{n-1}F_{2n-1}}{F^2_{n+r}\,F^2_{n-r-1}} = \frac{1}{F_{2r+1}} \sum_{j=1}^{2r+1} (-1)^j \Big/ F^2_j$$

SOME ADDITIONAL FORMULAS

Additional formulas together with an indication of the breakdown sums from which they were derived are given below.

$$\text{(7)}\qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{F_{2n+2}}{L^2_n\,L^2_{n+2}} = 8/45 \quad .$$

Derived from

$$\sum_{n=1}^{\infty} (F^2_n/L^2_n - F^2_{n+2}/L^2_{n+2})$$

$$\text{(8)}\qquad \sum_{n=1}^{\infty} \frac{1}{F_n\,F_{n+2}\,F_{n+3}} = 1/4$$

derived from

$$1/2 \sum_{n=1}^{\infty} \left[\frac{1}{F_nF_{n+1}F_{n+2}} - \frac{1}{F_{n+1}F_{n+2}F_{n+3}}\right]$$

$$\text{(9)}\qquad \sum_{n=1}^{\infty} \frac{F_{n+3}}{F_n\,F_{n+2}\,F_{n+4}\,F_{n+6}} = 17/480$$

derived from

$$\sum_{n=1}^{\infty} \left[\frac{1}{F_n\,F_{n+2}\,F_{n+4}} - \frac{1}{F_{n+2}\,F_{n+4}\,F_{n+6}}\right]$$

$$\text{(10)}\qquad \sum_{n=1}^{\infty} \frac{F_{4n+3}}{F_{2n}\,F_{2n+1}\,F_{2n+2}\,F_{2n+3}} = 1/2$$

derived from

$$\sum_{n=1}^{\infty} [1/(F_{2n}F_{2n+1}) - 1/(F_{2n+2}F_{2n+3})]$$

(11) $$\sum_{n=1}^{\infty} L_{n+2}/(F_n F_{n+4}) = 17/6$$

derived from

$$\sum_{n=1}^{\infty} (1/F_n - 1/F_{n+4})$$

(12) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} F_{2n+1}}{F_n^2 F_{n+1}^2} = 1$$

derived from

$$\sum_{n=1}^{\infty} (L_n^2/F_n^2 - L_{n+1}^2/F_{n+1}^2)$$

(13) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} L_{n+1}}{F_n F_{n+1} F_{n+2}} = 1$$

derived from

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[1/(F_n F_{n+1}) + 1/(F_{n+1} F_{n+2})\right]$$

(14) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{3n} L_{3n+3}} = \frac{3 - r}{40(1 + r)}$$

derived from

$$\sum_{n=1}^{\infty} (L_{3n-3}/L_{3n} - L_{3n}/L_{3n+3})$$

(15) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} F_{6n+3}}{F_{3n}^2 F_{3n+3}^2} = 1/8$$

derived from

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[1/F_{3n}^2 + 1/F_{3n+3}^2\right]$$

(16) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+5}} = \frac{150\, r^{-1} - 83}{150}$$

derived from

$$\sum_{n=1}^{\infty} (F_{n-1}/F_n - F_{n+4}/F_{n+5})$$

(17) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} F_{6n+3}}{F_{6n} F_{6n+6}} = 1/16$$

(18) $$\sum_{n=1}^{\infty} \frac{F_{2n+5}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5}} = 1/15$$

derived from $$\sum_{n=1}^{\infty} \left[\frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{1}{F_{n+2} F_{n+3} F_{n+4} F_{n+5}}\right]$$

(19) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n-1} F_{2n+3}} = 1/6$$

derived from

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{F_{2n-1} F_{2n+1}} + \frac{1}{F_{2n+1} F_{2n+3}}\right]$$

(20) $$\sum_{n=1}^{\infty} \frac{F_{2n}}{F_{n+2}^2 F_{n-2}^2} = 85/108$$

derived from

$$\sum_{n=1}^{\infty} \left[1/F_{n-2}^2 - 1/F_{n+2}^2\right]$$

CONCLUSION

Two main lines of development are open for continuing this research: (1) Building up a collection of formulas; (2) Finding additional methods for arriving at the summation of infinite Fibonacci-Lucas **series.**

Results, whether in the form of isolated formulas (with proof), or other more extensive developments should be reported to the Editor of the Fibonacci Quarterly.

REFERENCES

1. Problem H-10, proposed by R. L. Graham, FQ, April 1963, p. 53.
2. Problem B-9, proposed by R. L. Graham, FQ, April 1963, p. 85.
3. Problem B-19, proposed by L. Carlitz, FQ, Oct. 1963, p. 75.
4. Problem B-23, iii, proposed by S. L. Basin, FQ, Oct. 1963, p. 76.
5. Solution to Problem H-10, solution by L. Carlitz, FQ, Dec. 1963, p. 49.
6. Solution to B-9, solution by Francis D. Parker, FQ, Dec. 1963, p. 76.
7. Solution to B-19, solution to John H. Avila, FQ, Feb. 1964, p. 75.
8. Solution to B-23, iii, solution by J. L. Brown, Jr. , FQ, Feb. 1964, p. 79.
9. Problem H-56, proposed by L. Carlitz, FQ, Feb. 1965, p. 45.

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