ON THE DENSITY OF THE k-FREE INTEGERS

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Let T_k denote the set of k-free integers and let $T_k(n)$ be the number of such numbers not exceeding n. Then the Schnirelmann and asymptotic densities of T_k are defined by

$$(1) \qquad d(T_k) = \inf \frac{T_k(n)}{n}$$

and

$$(2) \qquad \delta(T_k) = \lim_{n\to\infty} \frac{T_k(n)}{n} = \frac{1}{\zeta(k)}$$

respectively, where $\zeta(s)$ is the Riemann zeta function. Our purpose is to summarize and extend the known results concerning the relationship between $d(T_k)$ and $\delta(T_k)$.

It has been shown by Rogers [1] that

$$(3) \qquad d(T_2) = \frac{53}{88} < \frac{6}{\pi^2} = \delta(T_2)$$

and it has been shown subsequently [2] that

$$(4) \qquad \delta(T_k) < d(T_{k+1}) \le \delta(T_{k+1}) .$$

The fact that $d(T_k) \le \delta(T_k)$ is an immediate consequence of (1) and (2). More recently, it has been shown by Stark [3] that

$$(5) \qquad d(T_k) < \delta(T_k) .$$

Combining (4) and (5), we have

$$d(T_k) < \delta(T_k) < d(T_{k+1}) , \tag{6}$$

i.e., the Schnirelmann and asymptotic densities of the k-free integers interlace.

The proofs of (3) and (4) and the second part of (2) are elementary while the proof of (5) is made to depend on what seems to be a much deeper result. Thus it would be very desirable to have a correspondingly simple proof of (5).

It is also easily shown [2] that

$$d(T_k) > 1 - \sum_p p^{-k}$$

from which it follows immediately that

$$d(T_k) > 2 - \zeta(k) . \tag{7}$$

We conclude this survey by showing that $d(T_{k+1})$ is much closer to $\delta(T_{k+1})$ than to $\delta(T_k)$.

To do this we define

$$\Delta(k) = \frac{\delta(T_{k+1}) - d(T_{k+1})}{\delta(T_{k+1}) - \delta(T_k)} \tag{8}$$

since the numerator and denominator in (8) are both positive, the following theorem yields the desired result.

<u>Theorem.</u> $\Delta(k) < 2^{-k}$.

<u>Proof.</u> By (2), (7) and (8) we have

$$\Delta(k - 1) < \frac{\frac{1}{\zeta(k)} - 2 + \zeta(k)}{\frac{1}{\zeta(k)} - \frac{1}{\zeta(k-1)}} = \frac{(\zeta(k) - 1)^2}{1 - \frac{\zeta(k)}{\zeta(k-1)}} .$$

But

$$\frac{\zeta(k-1)}{\zeta(k)} = \sum_{n=1}^{\infty} \phi(n) n^{-k} > \zeta(k),$$

where $\phi(n)$ is Euler's function. Hence

$$\Delta(k-1) < \zeta(k)\ (\zeta(k) - 1) \quad .$$

Since $\zeta(3) < 1.203$, the desired result follows from the trivial estimate

$$\zeta(k) < 1 + \frac{1}{2^k} + \frac{1}{3^k} + \int_3^{\infty} \frac{dx}{x^k} \le 1 + \frac{1}{2^k} + \frac{2}{3^k} \ .$$

It should be observed that this result also furnishes an alternative proof of the second inequality in (6).

REFERENCES

1. Kenneth Rogers, "The Schnirelmann Density of the Square Free Integers," Proc. Amer. Math. Soc. 15(1964), pp. 515-516.
2. R. L. Duncan, "The Schnirelmann Density of the k-Free Integers," Proc. Amer. Math. Soc. 16(1965), pp. 1090-1091.
3. H. M. Stark, "On the Asymptotic Density of the k-Free Integers," Proc. Amer. Math. Soc. 17(1966), pp. 1211-1214.

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