A BRAIN TEASER RELATED TO FIBONACCI NUMBERS

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A Swedish manufacturer of scientific instruments, LKB-Produkter AB, of Bromma, Sweden, has formed a tradition of sending out New Year's greetings in the form of mathematical brain teasers. A recent LKB brain teaser concerned a numerical problem encountered by the commander of a space ship and some members of his crew, which is composed of men from the Earth as well as men from Mars, Neptune and other planets.

THE PROBLEM

One day when calculating the distance which the ship had made, the Martian navigator, Lu, working with the decadic computer in the control room, obtained as a result a number the first five digits of which were 10112 and which had the property that if its last digit were moved to the first position a multiple of the number was formed.

Lu's complanetarian, Ku, tried to reconstruct Lu's number by manual calculation and was able to find a number beginning 10112 and also showing the desired property in regard of multiplication. However, Ku's number had only a little less than one-third as many digits as had Lu's.

A crew member of Neptune by name Elkeybub, who was known as a genius in mental calculation, was then called in to settle the dispute that Lu and Ku had got into because of the discrepancy between the numbers they had found. Elkeybub started the cells of his gray matter and soon came forth with his result: a number having one more digit than had Lu's, but otherwise fulfilling the same requirements as did Lu's and Ku's numbers.

The questions posed were: (1) why did not Lu, Ku, and Elkeybub get the same number, and (2) what numbers did they get?

THE GENERAL SOLUTION

It is easily shown that a number, N, having the property of being transformed into a multiple of itself when its last digit is moved to the first position has the form

$$N = \frac{c(B^n - 1)}{fB - 1} \quad , \tag{1}$$

where

- c = the last digit of N,
- B = the base of the number system,
- n = the number of digits of N
- f = the multiplicity factor ($= N'/N$, where N' is the number obtained by moving c to the first position).

By simple reasoning we find that the following relationships hold between the parameters:

$$2 \leq f \leq c < (B - 1) \tag{2}$$

(the case $f = 1$, or $N' = N$, is disregarded as being trivial).

That there exists for every set of (B, c, f) a value of n such that N is an integer can be shown by means of Fermat's theorem

$$x^{\phi(m)} \equiv 1 \bmod m, \tag{3}$$

where x and m are integers having no common factor, and $\phi(m)$ is the number of integers less than m and prime to it.

Now, if B and $(fB - 1)$ have no common factor, Eq. (1) will give an integer value of N for

$$n = \phi(fB - 1) \ . \tag{4}$$

It is immediately seen that b and $(fB - 1)$ can have no common factor, and Eq. (4) holds true.

If $(fB - 1)$ is prime, we get

$$n = fB - 2 \ . \tag{5}$$

If $(fB - 1)$ is composite, that is,

$$fB - 1 = d^q e^r \cdots, \quad d \neq 1, \ e \neq 0, \cdots \tag{6}$$

we get

$$n = \phi(fB - 1) = d^{q-1}(d - 1) \cdot e^{r-1}(e - 1) \cdots \quad . \tag{7}$$

It should be pointed out that the values of N obtained from Eq. (1) and either of Eqs. (5) or (7) will not necessarily be the smallest values possible, a factor of $\phi(fB - 1)$ sometimes being sufficient to produce an integral value of N.

This occurs, for (fB - 1) prime, with those values of (fB - 1) for which B is not a primitive root. For B = 10 (the decadic system) we have (fB - 1) = 19, 29, 39, $\cdots$, 89. Of these, 39, 49, and 69 are composite. Of the prime values, 19, 29 and 59 have 10 as primitive root:

$$\left.\begin{aligned} 10^{18} &\equiv 1 \bmod 19 \\ 10^{28} &\equiv 1 \bmod 29 \\ 10^{58} &\equiv 1 \bmod 59 \end{aligned}\right\}, \quad \text{or } 10^{p-1} \equiv 1 \bmod p \qquad \begin{aligned} n_{min} &= 18 \\ n_{min} &= 28 \\ n_{min} &= 58 \end{aligned} \quad .$$

For the remaining primes, 10 is not a primitive root, and we have

$$10^{13} \equiv 1 \bmod 79, \quad \text{or} \quad 10^{(p-1)/6} \equiv 1 \bmod p, \quad n_{min} = 13 ,$$

and

$$10^{44} \equiv 1 \bmod 89, \quad \text{or} \quad 10^{(p-1)/2} \equiv 1 \bmod p, \quad n_{min} = 44 .$$

In the case of the composite values of (fB - 1), their prime factors will decide whether $\phi(fB - 1)$ or a factor thereof will be the smallest n that satisfies Eq. (1). Here we get

fB - 1		ϕ(fB - 1)	n_{min}
$39 = 3 \cdot 13$	$10^1 \equiv 1 \bmod 3$ $10^6 \equiv 1 \bmod 13$	$2 \cdot 12 = 26$	6
$49 = 7^2$	$10^6 \equiv 1 \bmod 7$	$6 \cdot 7 = 42$	42
$69 = 3 \cdot 23$	$10^{22} \equiv 1 \bmod 23$	$2 \cdot 22 = 44$	22

Note that in the case $N(10,7,5)$, we have one of the very rare cases where c is a factor of $(fB - 1)$, that is, $(fB - 1)/c = 7$, and since $10^6 \equiv 1 \bmod 7$, $N(10,7,5)$ gives $n_{min} = 6$.

An interesting property of N is related to the following reasoning:

$$\begin{aligned} N_c &= N(B,c,f) = c(B^n - 1)/(fB - 1), \\ N_{c+1} &= N(B,c+1,f) = (c+1)(B^n - 1)/(fB - 1), \\ N_{c+1} - N_c &= (B^n - 1)/(fB - 1) = N_c/c . \end{aligned}$$

Also, if $N_1 = c(B^n - 1)/(fB - 1)$ is an integer, then

$$N_i = c(B^{in} - 1)/(fB - 1), \quad i = 1, 2, 3, \cdots,$$

are integers too, which means that any N gives rise to an infinite number of such numbers, formed by cyclic repetition of N.

METHODS OF CALCULATING $N(B,c,f)$

(1) By solving n from Eqs. (5) or (7) and inserting n, B, c, and f into Eq. (1).

(2) By dividing c by $(fB - 1)$ (neglect the decimal point!) until c appears as remainder, after which the quotient will repeat periodically, the period being equal to N.

(3) By a step-by-step multiplication

$$f \cdot N(B,c,f) = N' ,$$

bearing in mind that the digit in the second position of the multiplicand (N) shall be equal to the digit in the first position of the product (N').

(4) By a step-by-step division

$$N'/f = N(B,c,f) ,$$

which is the reciprocal to the multiplication method: the digit in the $(n-1)^{th}$ position of the dividend (N') shall be equal to the digit in the n^{th} position of the quotient, etc.

(5) By means of Fibonacci's numbers:

$$N = \sum_{i=1}^{\infty} a_i B^{n-1} - \sum_{i=1}^{\infty} a_i B^{-1} ,$$

where $a_i = a_{i-2} + (B - f)\, a_{i-1}$, $a_1 = 1$, $a_2 = c - f$.

The first term in Eq. (8) can be shown to be equal to

$$S = cB^n/(fB - 1) ,$$

and the second term,

$$S' = S/B^n = c/(fB - 1) .$$

Thus

$$N = S - S' = c(B^n - 1)/(fB - 1).$$

To illustrate method No. (5), we calculate Ku's number, N(6,5,5):

a_1		1
a_2	$= c - f$	0
a_3	$= a_1 + (B - f)a_2$	1
a_4	$= a_2 + (B - f)a_3$	1
a_5	$= a_3 + (B - f)a_4$	2
a_6	= etc.	3
a_7		5
a_8		12
a_9		21
a_{10}		3 3
a_{11}		54
a_{12}		131
a_{13}		22 5
a_{14}		4 00
a_{15}		1 025
a_{16}		142 5
a_{17}		24 54
a_{18}		4 323
a_{19}		1 122 1
a_{20}		155 44
a_{21}		31 205
$\cdots$		$\cdots$

$$101,\underbrace{124,045}_{N(6,5,5)},443,151,\cdots$$

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