UNIQUE REPRESENTATIONS OF INTEGERS AS SUMS OF DISTINCT LUCAS NUMBERS

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INTRODUCTION

The Lucas numbers, $\{L_n\}_0^\infty$, are defined by

$$L_0 = 2, \qquad L_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n$$

for $n \geq 0$. Then,

$$L_n = F_{n+1} + F_{n-1}$$

for $n \geq 0$, where

$$F_{-1} = 1, \quad F_0 = 0$$

and

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 1)$$

define the Fibonacci numbers. It is well-known that the Lucas numbers are "complete" [1] in the sense that every positive integer can be expressed as a sum of distinct Lucas numbers. In general, such representations are not unique; for example,

$$4 = L_3 = L_1 + L_2, \qquad 12 = L_1 + L_3 + L_4 = L_0 + L_2 + L_4,$$

etc. Our purpose in this paper is to show, by introducing constraints analogous to those used in obtaining unique expansions of integers in Fibonacci numbers, that unique representations in terms of Lucas numbers are also possible. We show, as one example, that every positive integer n has a unique representation of the form

(1) $$n = \sum_{0}^{\infty} \alpha_i L_i$$

where $\alpha_i = \alpha_i(n)$ is a binary digit (zero or one) for each $i \ge 0$ and the α_i satisfy the following constraints:

(2) $$\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \ge 0$$

(3) $$\alpha_0 \alpha_2 = 0 \ .$$

We recall that the constraint $\alpha_i \alpha_{i+1} = 0$, which precludes the use of two successive Lucas numbers in the representation, is essentially the same requirement that gives unique representations in Zeckendorf's theorem for Fibonacci expansions ([2], [3]). The additional condition $\alpha_0 \alpha_2 = 0$ reflects the particularity of the Lucas sequence.

REPRESENTATION THEOREMS

Before stating the main theorems, certain preliminary lemmas will prove useful.

Lemma 1.

$$L_n - 1 = L_{n-1} + L_{n-3} + \cdots + L_{1,2}(n)$$

for $n \ge 2$,

where

$$L_{1,2}(n) = \begin{cases} 2L_1 & \text{if } n \text{ is even} \\ L_2 & \text{if } n \text{ is odd} \ . \end{cases}$$

Proof. By induction, one easily proves

$$L_{2n+1} - 1 = L_{2n} + L_{2n-2} + \cdots + L_4 + L_2 \quad (n \geq 1)$$

$$L_{2n} - 1 = L_{2n-1} + L_{2n-3} + \cdots + L_3 + 2L_1 \quad (n \geq 1) .$$

The Lemma statement combines these two identities.

Lemma 2.

$$L_{n+2} = 1 + \sum_{i=0}^{n} L_i \quad \text{for} \quad n \geq 0 .$$

Proof. Induction.

Lemma 3. Let

$$n = \sum_0^\infty \alpha_i L_i ,$$

where each α_i is a binary digit such that

i) $\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 0$

ii) $\alpha_0 \alpha_2 = 0$.

Such a representation for n is unique.

Proof. Assume n has a competing representation,

$$n = \sum_0^\infty \gamma_i L_i$$

with γ_i binary, $\gamma_i \gamma_{i+1} = 0$ for $i \geq 0$ and $\gamma_0 \gamma_2 = 0$. Assume, for a proof by contradiction, that the two representations are not identical, that is,

$$\sum_0^{\infty} |\gamma_i - \alpha_i| \neq 0 .$$

Then, let k be the largest value of i such that $\alpha_i \neq \gamma_i$. Clearly $k \geq 2$, and since $\alpha_k \neq \gamma_k$, we may assume without loss of generality that $\alpha_k = 1$, $\gamma_k = 0$. It follows that, for some $m \leq n$,

$$m = \sum_0^k \alpha_i L_i = \sum_0^{k-1} \gamma_i L_i ,$$

with $\alpha_k = 1$. Then

$$\sum_0^k \alpha_i L_i \geq L_k ,$$

while from the coefficient constraints on the $\{\gamma_i\}$,

$$\sum_0^{k-1} \gamma_i L_i \leq L_{k-1} + L_{k-3} + \cdots + L_{1,2}(k) = L_k - 1 ,$$

the last equality from Lemma 1. Thus $m \geq L_k$ while $m \leq L_k - 1$, a contradiction.

Lemma 4. Let

$$n = \sum_0^k \beta_i L_i \quad (k \geq 2) ,$$

where each β_i is a binary digit such that

i) $\beta_i + \beta_{i+1} \neq 0$ for $0 \leq i \leq k-2$

ii) $$\beta_0 + \beta_2 \neq 0$$
iii) $$\beta_k = 1 .$$

Such a representation for n is unique.

Proof. Assume n has two representations in the given form; that is,

$$n = \sum_{i=0}^{k} \beta_i L_i = \sum_{i=0}^{m} \gamma_i L_i , \tag{4}$$

where β_i and γ_i are binary digits satisfying

$$\beta_k = \gamma_m = 1, \; \beta_i + \beta_{i+1} \neq 0$$

for $0 \leq i \leq k - 2$,

$$\beta_0 + \beta_2 \neq 0, \; \gamma_i + \gamma_{i+1} \neq 0$$

for $0 \leq i \leq m - 2$,

$$\gamma_0 + \gamma_2 \neq 0 .$$

Without loss of generality, we take $m \geq k \geq 2$. If $m > k$, then the right-hand representation in (4), together with the coefficient constraints, implies

$$n \geq \begin{cases} L_m + L_{m-2} + \cdots + L_2 + L_1 = L_{m+1} \geq L_{k+2} & (m \text{ even}) \\ L_m + L_{m-2} + \cdots + L_3 + L_1 + L_0 = L_{m+1} \geq L_{k+2} & (m \text{ odd}) . \end{cases}$$

But

$$n = \sum_{i=0}^{k} \beta_i L_i \leq \sum_{i=0}^{k} L_i = L_{k+2} - 1 ,$$

a contradiction. Hence $m = k$ in (4); that is,

$$n = \sum_0^k \beta_i L_i = \sum_0^k \gamma_i L_i ,$$

or equivalently,

$$\sum_0^k (1 - \beta_i) L_i = \sum_0^k (1 - \gamma_i) L_i .$$

If we now define $\alpha_i = 1 - \beta_i$ and $\delta_i = 1 - \gamma_i$ for $0 \le i \le k$ and $\alpha_i = \delta_i = 0$ for $i \ge k$, then

$$\sum_0^\infty \alpha_i L_i = \sum_0^\infty \delta_i L_i ,$$

with α_i, δ_i binary digits satisfying

$$\alpha_i \alpha_{i+1} = \delta_i \delta_{i+1} = 0$$

for all $i \ge 0$ and

$$\alpha_0 \alpha_2 = \delta_0 \delta_2 = 0 .$$

By Lemma 3, $\alpha_i = \delta_i$ for $i \ge 0$ and thus $\beta_i = \gamma_i$ for $0 \le i \le k$, implying uniqueness of the representation.

Theorem 1. Let n be a nonnegative integer satisfying $0 \le n < L_k$ for some $k \ge 1$. Then

(5) $$n = \sum_0^{k-1} \alpha_i L_i$$

with α_i binary digits satisfying

i) $\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \ge 0$

ii) $\alpha_0 \alpha_2 = 0$.

Further, the representation of n in this form is unique. [If $k - 1 < 2$ in (5), we define $\alpha_2 = 0$ so that ii) is automatically satisfied.]

Proof. Uniqueness follows from Lemma 3. It remains to show such a representation exists. For a proof by induction on the index k, we verify directly that the theorem holds for $k = 1$ and $k = 2$. Now, assume as an induction hypothesis that the theorem holds for all $k \le k_0$ where $k_0 \ge 2$. To show the theorem holds for $k_0 + 1$, it suffices to consider an arbitrary integer n satisfying

$$L_{k_0} \le n \le L_{k_0+1} .$$

Then

$$0 \le n - L_{k_0} < L_{k_0+1} - L_{k_0} = L_{k_0-1} .$$

By the induction hypothesis, there exist binary coefficients γ_i such that

$$n - L_{k_0} = \sum_0^{k_0-2} \gamma_i L_i$$

with

$$\gamma_i \gamma_{i+1} = 0 \text{ for } i \ge 0, \quad \gamma_0 \gamma_2 = 0 .$$

Then

$$n = \sum_0^{k_0} \gamma_i L_i$$

where

$$\gamma_{k_0-1} = 0, \quad \gamma_{k_0} = 1,$$

so that n is representable in the required form with the given coefficient constraints. q. e. d.

Theorem 2. Let n be a positive integer satisfying

$$\sum_0^{k-1} L_i < n \le \sum_0^k L_i$$

for some $k \ge 2$. Then

$$n = \sum_0^k \beta_i L_i$$

with β_i binary coefficients satisfying

i) $\beta_i + \beta_{i+1} \ne 0$ for $0 \le i \le k - 2$

ii) $\beta_0 + \beta_2 \ne 0$

iii) $\beta_k = 1$.

Further, the representation of n in this form is unique.

Proof. Again, uniqueness is a consequence of Lemma 4. To establish the representation, note that

$$\sum_0^{k-1} L_i < n \le \sum_0^k L_i$$

implies

$$0 \le \sum_0^k L_i - n < \sum_0^k L_i - \sum_0^{k-1} L_i = L_k .$$

By Theorem 1, the integer

$$\sum_0^k L_i - n$$

has a representation

$$\sum_0^k L_i - n = \sum_0^{k-1} \alpha_i L_i ,$$

where the binary coefficients α_i satisfy $\alpha_i\alpha_{i+1} = 0$ for

$$0 \leq i \leq k - 2, \quad \alpha_0\alpha_2 = 0 .$$

Then

$$n = L_k + \sum_0^{k-1} (1 - \alpha_i)L_i = \sum_0^k (1 - \alpha_i)L_i ,$$

where $\alpha_k = 0$, and the theorem follows on recognizing $\beta_i = 1 - \alpha_i$ $(0 \leq i \leq k)$ as binary coefficients satisfying

$$\beta_i + \beta_{i+1} \neq 0$$

for $0 \leq i \leq k - 2$, $\beta_0 + \beta_2 \neq 0$ and $\beta_k = 1$. q. e. d.

Theorem 2 thus guarantees the representation for all positive integers ≥ 4. Representations for the positive integers 1, 2, 3 are immediate, namely

$$1 = 0 \cdot L_0 + 1 \cdot L_1 , \quad 2 = 1 \cdot L_0, \quad 3 = 1 \cdot L_0 + 1 \cdot L_1 .$$

The constraint $\beta_0 + \beta_2 \neq 0$ is assumed not to be enforced in these three cases where the largest Lucas number appearing in the expansion is less than $L_2 = 3$.

Theorem 2 is a dual to Theorem 1 and corresponds to the dual of the Zeckendorf theorem for Fibonacci numbers [4].

REFERENCES

1. J. L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, Vol. 68, No. 6, June-July, 1961, pp. 557-560.
2. C. G. Lekkerkerker, "Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci," Simon Stevin, Vol. 29, 1951-52, pp. 190-195.
3. J. L. Brown, Jr., "Zeckendorf's Theorem and Some Applications," The Fibonacci Quarterly, Vol. 2, No. 3, October, 1964, pp. 163-168.
4. J. L. Brown, Jr., "A New Characterization of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 3, No. 1, February 1965, pp. 1-8.

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ASSOCIATION MEETING

The Fibonacci Association held its Fall Meeting on October 18th at San Jose State College. Following was the Program:

MORNING SESSION

9:30 a.m.	SOCIAL GATHERING
10:00 — 10:45	TEST FOR THE PRIMALITY OF MERSENNE NUMBERS Douglas Lind, Stanford University
10:45 — 11:30	WEB SEQUENCES George Ledin, Jr., University of San Francisco
11:30 — 12 Noon	OPPORTUNITY FOR GENERAL DISCUSSION

AFTERNOON SESSION

1:15 — 2:00	FIBONACCI AND RELATED SERIES IN COMBINATORICS Prof. D. H. Lehmer, University of Calif., Berkeley
2:00 — 2:45	MARKOV-FIBONACCI RELATIONS Prof. Gene Gale, San Jose State College
2:45 — 3:30	IT'S GENERALIZED! WHAT'S NEXT? Prof. V. C. Harris, San Diego State College

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