Π IN THE FORM OF A CONTINUED FRACTION WITH INFINITE TERMS

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$$\frac{\Pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\;\cdots \qquad \text{(Wallis)}$$

$$\therefore\ \Pi = \frac{2}{1}\cdot\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\;\cdots\quad ,$$

for which the successive products, as $n \to \infty$, are:

$$\frac{4}{1}, \frac{8}{3}, \frac{32}{9}, \frac{128}{45}, \frac{768}{225}, \cdots \frac{p_n}{q_n} \cdots \quad ,$$

for which the ordinal numbers are $1, 2, 3, \cdots, n, \cdots$.

These products are identical with the convergents, after the first two, of the following continued fraction with infinite terms, for which the ordinal numbers are $-1, 0, 1, 2, 3, \cdots, n$.

$$\Pi = \cfrac{4}{1 + \cfrac{1}{1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{2\cdot 3}{1 + \cfrac{3\cdot 4}{1 + \cdots \cfrac{(n-1)n}{1\cdots}}}}}}} \; .$$

These C. F. convergents are, when spun out in ordinal succession:

$$4, \frac{4}{2}, \frac{4}{1}, \frac{8}{3}, \frac{32}{9}, \frac{128}{45}, \frac{768}{225}, \cdots \frac{p_n}{q_n} \cdots \quad ,$$

The series corresponding to the infinite C. F. is:

$$\Pi = 4 - 2 + 2 - \frac{4}{3} + \frac{8}{9} - \frac{32}{45} + \frac{128}{225} - \frac{768}{1575} + \cdots + (-1)^{n-1}\frac{p_{n-1}}{q_n}\cdots$$

the p_{n-1} and q_n being the products as found above in the successive products for Π.

(The author acknowledges with appreciation the help of Lavar Rigby, Instructor for Computers at Ventura College, who checked the convergence trend of the subject continued fraction for Π on an IBM 1620.)

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ELEMENTARY PROBLEMS AND SOLUTIONS

B-153 Proposed by Klaus-Gunther Recke, Gottingen, Germany.

Prove that

$$F_1F_3 + F_2F_6 + F_3F_9 + \cdots + F_nF_{3n} = F_nF_{n+1}F_{2n+1}\,.$$

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

Since $F_1F_3 = F_1F_2F_3$, we need only show that when we add one to n, the increase on the left side of the equation is the same as that on the right. The increase on the left side is F_nF_{3n}; and, using the solution to B-152 with $m = 2n$,

$$\begin{aligned} F_nF_{3n} &= F_nF_{2n+n} \\ &= F_n(F_{2n+1}F_{n+1} - F_{2n-1}F_{n-1}) \\ &= F_nF_{n+1}F_{2n+1} - F_{n-1}F_nF_{2n-1} \end{aligned}$$

which is just the increase on the right side of the equation.

Also solved by Clyde A. Bridger, Herta T. Freitag, Serge Hamelin (Canada), John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, C. C. Yalavigi (Mercara, India), David Zeitlin, and the Proposer.

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