REMARK ON A THEOREM BY WAKSMAN

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Let Q denote the set of primes $Q = Q^\star \cup \{1\}$, Z the nonnegative integers and $V = \{K : Q^\star \le S_K\}$, where $S_K = \{m = Kn + p;\ n \in Z$ and $p = 1$ or $p \in Q$ such that $p \nmid K,\ p < K\} \cup \{p \in Q : p|K\}$. Let $U = \{k : k \in Z$ and each of the $\varphi(k)$ integers $1 = a_1 < a_2 < \cdots < a_{\varphi(k)}$ not greater than k and relatively prime to k, is a member of $Q^\star\}$. We note that $a_2 \in Q$ if $k > 2$.

A. Waksman [1] has shown (with the aid of a computer search) that $V = \{2, 3, 4, 6, 8, 12, 18, 24, 30\}$. Trivially, 1 must also be a member of V. We shall show that U = V. It is known that U consists of the integers given above [2, p. 62].

Let $0 < t \in Z$ and let $1 = a_1 < a_2 < \cdots < a_{\varphi(t)}$ be the integers not greater than t and relatively prime to t.

(i) We prove first that $U \subseteq V$. If $t \in U$ (so that $a_i \in Q^\star$) then every positive integer relatively prime to t is a member of the set

$$R = \{tn + a_i : n \in Z,\quad i = 1, 2, \cdots, \varphi(t)\}.$$

Now $1 \in R$ and if q is a prime, then either $q|t$ or $q \in R$. Thus $Q^\star \le S_t$ and $t \in V$.

(ii) We show now that $V \subseteq U$ (using, in part, a method of Waksman). It is immediate that 1 and $2 \notin V \cap Q$. If $2 < t \in V$ then by the Dirichlet theorem, there is a prime q such that $q = a_2^2 \pmod t$. Since $q \in S_t$ and $q \nmid t$ there is a prime $p < t$ such that $q \equiv p \pmod t$. Thus $p \equiv a_2^2 \pmod t$. If $a_2^2 < t$ then $t \,|\, |a_2^2 - p| < t$, which implies $p = a_2^2$, a contradication. Thus $a_2^2 \ge t$. If one of $a_i \notin Q$ $(i = 3, \cdots, \varphi(t))$, then $a_i \ge a_2^2 \ge t$, a contradiction. Thus $a_i \in Q^\star$ $(i = 1, 2, \cdots, \varphi(t))$, and $t \in U$.

REFERENCES

1. A. Waksman, "On the Distribution of Primes," American Mathematical Monthly, 75 (1968), pp. 764-765.
2. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Chelsea, New York, 1953.

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