ON THE COMPLETENESS OF THE LUCAS SEQUENCE

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It is well known* that the Lucas sequence

$$L_0,\ L_1,\ L_2,\ \cdots = 2,\ 1,\ 3,\ \cdots$$

is complete. It is easy to see that if $0 \le m < n$, the integer $L_{n+1} - 1$ can't be represented as a sum of distinct L_i with $i \ne m, n$. Thus $\{L_j\}$ is not complete after the removal of two arbitrary terms L_m, L_n. We will also show that the sequence is complete after the removal of any one term L_n with $n \ge 2$.

Let N be a positive integer. It is well known that N is a (maximal) sum of L_i's, that is,

$$(1)\qquad N = L_{i_1} + L_{i_2} + \cdots + L_{i_\beta} \quad \text{with} \begin{cases} i_1 \ge 0 \text{ and} \\ i_{\nu+1} - i_\nu \ge 2 \text{ for } 1 \le \nu < \beta . \end{cases}$$

We suppose L_n is one of the terms in the representation (1), for otherwise we have nothing to show, say $n = i_\alpha \le i_\beta$. Then

$$(2)\qquad M = L_{i_1} + L_{i_2} + \cdots + L_{i_\alpha} \le L_n + L_{n-2} + \cdots + L_k + L_0$$
$$= \begin{cases} L_{n+1} + 1 \text{ and } k = 2 \text{ if } n \text{ is even,} \\ L_{n+1} - 1 \text{ and } k = 3 \text{ if } n \text{ is odd .} \end{cases}$$

If $M = L_{n+1} + 1$, we replace the sum (2) for M by $L_1 + L_{n+1}$ in (1). If $M = L_{n+1}$ we replace the sum (2) for M by L_{n+1} in (1). Observe that L_{n+1} does not appear in (1). If $M \le L_{n+1} - 1$, we can re-represent it as a sum of distinct terms L_i with $0 \le i \le n - 1$, and so we are through in this final case.

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*V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Co., Boston, 1969.

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