ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-172 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Let $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for $n = 0, 1, \cdots$. Show that

$$F_{n+2}^3 = F_n^3 + F_{n+1}^3 + 3F_nF_{n+1}F_{n+2} .$$

B-173 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Show that

$$F_{3n} = F_{n+2}^3 - F_{n-1}^3 - 3F_nF_{n+1}F_{n+2} .$$

B-174 Proposed by Mel Most, Ridgefield Park, New Jersey.

Let a be a non-negative integer. Show that in the sequence

$$2F_{a+1},\quad 2^2F_{a+2},\quad 2^3F_{a+3},\quad \cdots$$

all differences between successive terms must end in the same digit.

B-175 Composed from the Solution by David Zeitlin to B-155.

Let r and q be constants and let $U_0 = 0$, $U_1 = 1$, $U_{n+2} = rU_{n+1} - qU_n$. Show that

$$U_{n+a}U_{n+b} - U_{n+a+b}U_n = q^nU_aU_b .$$

B-176 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. M.

Let $\begin{bmatrix} n \\ r \end{bmatrix}$ denote the Fibonomial Coefficient

$$F_nF_{n-1}\cdots F_{n-r+1}/F_1F_2\cdots F_r .$$

Show that

$$F_n^3 = \begin{bmatrix} n+2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} n+1 \\ 3 \end{bmatrix} - \begin{bmatrix} n \\ 3 \end{bmatrix} .$$

B-177 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. M.

Using the notation of B-176, show that

$$F_n^4 = \begin{bmatrix} n+3 \\ 4 \end{bmatrix} - a\begin{bmatrix} n+2 \\ 4 \end{bmatrix} - a\begin{bmatrix} n+1 \\ 4 \end{bmatrix} + \begin{bmatrix} n \\ 4 \end{bmatrix} ,$$

for some integer a and find a.

SOLUTIONS

A VERY MAGIC SQUARE

B-154 Proposed by S. H. L. Kung, Jacksonville University, Jacksonville, Fla.

What is special about the following "magic" square?

$$\begin{bmatrix} 11 & 2 & 14 & 19 & 21 \\ 8 & 13 & 3 & 22 & 1 \\ 20 & 17 & 15 & 6 & 9 \\ 7 & 24 & 18 & 10 & 12 \\ 25 & 5 & 23 & 16 & 4 \end{bmatrix}$$

Solution by the Proposer.

(a) The sum of all the numbers contained in a row, or in a column, or in a diagonal is a prime.

(b) The sum of the squares of all numbers contained in a row, column, or diagonal is also a prime.

Solvers Guy A. Guillottee (Quebec, Canada) and Michael Yoder listed observation (a) above.

A PELL NUMBERS IDENTITY

B-155 Composite of proposals by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada, and Carol Anne Vespe, University of New Mexico, Albuquerque, New Mexico.

Let the n^{th} Pell number be defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that

$$P_{n+a}P_{n+b} - P_{n+a+b}P_n = (-1)^n P_a P_b \ .$$

Solution by Wray G. Brady, University of Bridgeport, Bridgeport, Conn.

One finds that

$$P_n = \frac{r^n - s^n}{2\sqrt{2}} \ ,$$

where $r = 1 + \sqrt{2}$, $s = 1 - \sqrt{2}$, and $rs = -1$. Then

$$\begin{aligned} 8(P_{n+a}P_{n+b} - P_{n+a+b}P_n) &= r^{2n+ab} - r^{n+b}s^{n+a} - r^{n+a}s^{n+b} + s^{2n+a+b} \\ &\quad - (r^{2n+a+b} - r^{n+a+b}s^n - r^n s^{n+a+b} + s^{2n+a+b}) \\ &= (-1)^n(r^{a+b} + s^{a+b} - r^a s^b - r^b s^a) \\ &= (-1)^n(r^a - s^a)(r^b - s^b) = 8P_aP_b(-1)^n \ , \end{aligned}$$

and the desired result follows.

EDITORIAL NOTE: Let $f_a(x) = 0$, $f_1(x) = 1$ and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$; then $f_n(2) = P_n$.

Also solved by Herta T. Freitag, Guy A. Guillottee (Quebec, Canada), Serge Hamelin (Quebec, Canada), Bruce W. King, C. B. A. Peck, A. G. Shannon (Boroko, T.P.N.G.), Gregory Wulczyn, Michael Yoder, David Zeitlin, and the Proposers.

PERIODIC REMAINDERS

B-156 Proposed by Allan Scott, Phoenix, Arizona.

Let F_n be the n^{th} Fibonacci number, $G_n = F_{4n} - 2n$, and H_n be the remainder when G_n is divided by 10.

(a) Show that the sequence $H_2, H_3, H_4, \cdots$ is periodic and find the repeating block.

(b) The last two digits of G_9 and G_{14} give Fibonacci numbers 34 and 89, respectively. Are there any other cases?

Solution by Herta T. Freitag, Hollins, Virginia.

(a) Since $F_1 \equiv 1 \pmod{10}, \cdots, F_{59} \equiv 1 \pmod{10}$, $F_{60} \equiv 0$ we have $F_1 \equiv F_{1+4\cdot 15k} \pmod{10}$, where k is any positive Also $G_n = F_{4n} - 2n \equiv H_n \pmod{10}$ and $2n \equiv 2, 4, 6, 8, 0 \pmod{10}$ for $n \equiv 1, 2, 3, 4, 0 \pmod 5$. Thus the repeating block of 15 terms $H_1, H_2, \cdots, H_{15}$ is 1, 7, 8, 9, 5, 6, 7, 3, 4, 5, 1, 2, 3, 9, 0 and $H_i = H_{i+15k}$ for integers i and k.

(b) By studying the corresponding pattern modulo 100 we detect another periodicity cycle such that all Fibonacci numbers smaller than 100 must occur within the last two digits of G_n provided $1 \le n \le 150$. More explicitly, we indicate the G corresponding to a given F in the following table:

Subscript on F	1 or 2	3	4	5	6
Subscript on G	1 or 26	87	118 or 143	55 or 80	48

Subscript on F	7	8	9	10	11
Subscript on G	103 or 128	121 or 146	9	5 or 130	14 or 139

Also solved by Serge Hamelin (Quebec, Canada), C. B. A. Peck, and the Proposer.

A TELESCOPING SUM

B-157 Proposed by Klaus Gunther Recke, University of Gottingen, Germany.

Let F_n be the n^{th} Fibonacci number and $\{g_n\}$ any sequence. Show that

$$\sum_{k=1}^{n} (g_{k+2} + g_{k+1} - g_k)F_k = g_{n+2}F_n + g_{n+1}F_{n+1} - g_1 .$$

Solution by John E. Homer, Jr., Union Carbide Corporation, Chicago, Illinois.

The sum is equivalent to

$$\sum_{k=3}^{n} g_k(F_{k-2} + F_{k-1} - F_k) + g_{n+2}F_n + g_{n+1}(F_n + F_{n-1}) + g_2F_1 - g_2F_2 - g_1F_1$$

$$= g_{n+2}F_n + g_{n+1}F_{n+1} - g_1,$$

since

$$F_{k-2} + F_{k-1} - F_k = 0 .$$

Also solved by Wray G. Brady, Herta T. Freitag, Serge Hamelin (Quebec, Canada), Bruce W. King, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T.P.N.G.), Michael Yoder, David Zeitlin, and the Proposer.

ANOTHER TELESCOPING SUM

B-158 Proposed by Klaus Günther Recke, University of Gottingen, Germany.

Show that

$$\sum_{k=1}^{n} (kF_k)^2 = [(n^2 + n + 2)F_{n+2}^2 - (n^2 + 3n + 2)F_{n+1}^2 - (n^2 + 3n + 4)F_n^2]/2 .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let H_n satisfy $H_{n+2} = H_{n+1} + H_n$. Noting that

$$H_{n+3}^2 = 2H_{n+2}^2 + 2H_{n+1}^2 - H_n^2 ,$$

it is easy to show, using mathematical induction, that

$$2\sum_{k=1}^{n} k^2H_k^2 = (n^2 + n + 2)H_{n+2}^2 - (n^2 + 3n + 2)H_{n+1}^2 - (n^2 + 3n + 4)H_n^2 + C,$$

where

$$C = 6H_1^2 + 2H_2^2 - 8H_1H_2 .$$

We note that $C = 0$ when $H_k = F_k$ or $H_k = L_k$.

Also solved by Herta T. Freitag, Serge Hamelin (Quebec, Canada), John E. Homer, Jr., Bruce W. King, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T.P.N.G.), Michael Yoder, and the Proposer.

THE EULER TOTIENT

B-159 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let T_n be the n^{th} triangular number $n(n+1)/2$ and let $\varphi(n)$ be the Euler totient. Show that

$$\varphi(n) \mid \varphi(T_n)$$

for $n = 1, 2, \cdots$.

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, N. M.

We assume it is known that $\varphi(ab) = \varphi(a)\varphi(b)$ if $(a,b) = 1$; $\varphi(n)$ is even if $n > 2$; and $\varphi(2^k)$ if $k \geq 1$. Let $n = 2^k s$, where $2 \nmid s$. The proof is in three cases.

Case 1. $k = 0$. Then

$$\varphi(T_n) = \varphi\left(n \cdot \frac{n+1}{2}\right) = \varphi(n)\varphi\left(\frac{n+1}{2}\right),$$

since

$$\left(n, \frac{n+1}{2}\right) = 1 .$$

Case 2. $k = 1$. Note that

$$\varphi(n) = \varphi(2)\varphi(s) = \varphi(s) = \varphi\left(\frac{n}{2}\right),$$

so

$$\varphi(T_n) = \varphi\left(\frac{n}{2}\right)\varphi(n+1) = \varphi(n)\varphi(n+1).$$

Case 3. $k > 1$. Now

$$\varphi(n) = \varphi(2^k)\varphi(s) = 2^{k-1}\varphi(s),$$

and

$$\varphi\left(\frac{n}{2}\right) = 2^{k-2}\varphi(s).$$

Also we obviously have $n + 1 > 2$; so let $\varphi(n + 1) = 2m$, where m is an integer. Then

$$\varphi(T_n) = \varphi\left(\frac{n}{2}\right)\varphi(n+1) = 2^{k-2}\varphi(s)2m = m \cdot 2^{k-1}\varphi(s) = m\varphi(n).$$

Also solved by Herta T. Freitag, Guy A. Guillottee (Canada), Serge Hamelin (Canada), Douglas Lind (England), C. B. A. Peck, Gregory Wulczyn, and the Proposer.

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or

$$491 = x(x+1) + y(y+1) + z(z+1).$$

This is impossible, since $x(x+1)$, $y(y+1)$, and $z(z+1)$ are all even.

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